ENG ME 740:

Exercises (Set 6)

1. Write down the equations describing the dynamics of the double pendulum depicted in the figure with the force of gravity acting vertically downward. Assume massless links with all mass concentrated at points in the elbow joint and tip.

![Diagram of double pendulum](image)

Figure 1:

2. Let $A$ be a matrix with $m$ rows and $n$ columns, $m < n$. Let $W$ be an $n \times n$ positive definite matrix. Show that for a given $m$-vector $x$ on to the linear equation $Ax = y$

that minimizes the norm square $\|x\|_W^2 = x^T W^{-1} x$ is given in terms of the weighted pseudo-inverse by

$$x_0 = W A^T (A W A^T)^{-1} y.$$
1. Position $m_1$:

\[
\begin{pmatrix}
  r_1 s_1 \\
  -r_1 c_1
\end{pmatrix}
\]

Velocity $m_1$:

\[
\begin{pmatrix}
  r_1 c_1 \\
  r_1 s_1
\end{pmatrix}
\frac{\dot{\theta}_1}{}
\]

\[
\begin{pmatrix}
  r_1 s_1 \\
  -r_1 c_1
\end{pmatrix}
\frac{\dot{\theta}_1}{} + \begin{pmatrix}
  r_2 c_2 \\
  r_2 s_2
\end{pmatrix}
\frac{\dot{\theta}_2}{}
\]

K.E.

\[
= \frac{1}{2} m_1 r_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left[ (r_1 c_1 \dot{\theta}_1 + r_2 c_2 \dot{\theta}_2)^2 + (r_1 s_1 \dot{\theta}_1 + r_2 s_2 \dot{\theta}_2)^2 \right]
\]

\[
= \frac{1}{2} (m_1 + m_2) r_1^2 \dot{\theta}_1^2 + m_2 r_1 r_2 \cos(\theta_1 - \theta_2) \dot{\theta}_1 \dot{\theta}_2 + \frac{1}{2} m_2 r_2^2 \dot{\theta}_2^2
\]

P.E.

\[
= -(m_1 + m_2) g r_1 c_1 - m_2 g r_2 c_2
\]

\[
L = T - V = \text{K.E.} - \text{P.E.}
\]

\[
= \frac{1}{2} (m_1 + m_2) r_1^2 \dot{\theta}_1^2 + m_2 r_1 r_2 \cos(\theta_1 - \theta_2) \dot{\theta}_1 \dot{\theta}_2 + \frac{1}{2} m_2 r_2^2 \dot{\theta}_2^2
\]

\[
+ (m_1 + m_2) g r_1 c_1 + m_2 g r_2 c_2
\]
Equations of motion: \[ \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} = 0 \]

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} = (m_1 + m_2) r_1^2 \ddot{\theta}_1 + m_2 r_1 r_2 \cos(\theta_1 - \theta_2) \ddot{\theta}_2 
+ m_2 r_1^2 \sin(\theta_1 - \theta_2) \dot{\theta}_1^2 + (m_1 + m_2) r_1 \sin \theta_1 \eta 
\]

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} = m_2 r_1 r_2 \cos(\theta_1 - \theta_2) \ddot{\theta}_1 + m_2 r_2^2 \ddot{\theta}_2 
- m_2 r_1 r_2 \sin(\theta_1 - \theta_2) \dot{\theta}_1^2 + m_2 r_2 \sin \theta_2 \eta \]

2. The solution follows a line of argument similar to what was done in class for the special case \( W=I \):

We wish to minimize \( x^T W^{-1} x \) subject to the constraint that \( Ax=y \).

We shall find critical points of \( \frac{1}{2} x^T W^{-1} x + \lambda^T (A x - y) \).
WHERE $\lambda$ is an $m$-vector of Lagrange multipliers.

**Critical Point Equation:**

$$W^{-1}x + A^T\lambda = 0$$

$$\iff x + W A \lambda = 0$$

**Multiply both sides by $A$:**

$$A x + A W A^T \lambda = 0$$

$$\iff y + A W A^T \lambda = 0.$$  

**Claim:** $A W A^T$ is invertible. Otherwise, arguing as was done in class, there would be a non-zero vector $v$ s.t.

$$A W A^T v = 0$$
\[ \Rightarrow \quad \mathbf{v}^T \mathbf{A} \mathbf{W} \mathbf{A}^T \mathbf{v} = 0 \]

\[ \Leftrightarrow \quad \mathbf{A}^T \mathbf{v} = 0 \]

which would contradict \( \mathbf{A} \) having rank \( m \).

This proves the claim. Hence, we may write

\[ \lambda = - (\mathbf{A} \mathbf{W} \mathbf{A}^T)^{-1} \mathbf{y}. \]

Plugging this into the original critical point equation, we get

\[ \mathbf{x} = \mathbf{W} \mathbf{A} \mathbf{T} (\mathbf{A} \mathbf{W} \mathbf{A}^T)^{-1} \mathbf{y}. \]