Lie Groups I

Review

- Geometric Relationships in the plane $\mathbb{R}^2$ and space $\mathbb{R}^3$
- Group Theory
- Matrix exponentials

**Definition:** A subgroup of the group of all $n \times n$ invertible matrices is called a **Lie Group** (matrix Lie Group) if it is also a closed sub-manifold (you can do calculus on it).

**Example:** The “winding line” that is dense on the 2-torus is not a Lie Group.

**Definition:** A vector space $V$ (over $\mathbb{R}$) is a **Lie Algebra** if in addition to the vector space structure there is defined a binary operation:

$\left[ \cdot, \cdot \right] : V \times V \rightarrow V$ called “Lie Bracket”

satisfying the following properties:

1. **Bilinearity property**
   
   \[ a_1 v_1 + a_2 v_2, w \] = \[ a_1 v_1, w \] + \[ a_2 v_2, w \] for all \( v_1, v_2, w \in V, \ a_1, a_2, \in \mathbb{R} \)

2. **Skew symmetry**
   
   \[ v, w \] = -\[ w, v \] for all \( v, w \in V \)

3. **Jacobi Identity**
   
   \[ [v, [w, z]] + [w, [z, v]] + [z, [v, w]] = 0 \] for all \( v, w, z \in V \)

**Three 3 \times 3 Examples**

1. Vector cross product in $\mathbb{R}^3$, \( v, w, \in \mathbb{R}^3 \), \( [v, w] = v \times w \in \mathbb{R}^3 \)

2. $so(3)$, vector space of $3 \times 3$ skew-symmetric matrices, (with $[A, B] = AB - BA$)

3. $sl(2)$, set of all $2 \times 2$ matrices with trace = 0, (with $[A, B] = AB - BA$)

Given a matrix Lie Group $G$, we wish to study the tangent space at the identity. Let $S(t)$ be a curve in $G$ such that $S(0) = I$, \( S'(0) = A \) (element in the tangent space). Let $R \in G$. Then $T(t) = R S(t) R^{-1}$ and $T(0) = I$. Hence $T'(0) = R S'(0) R^{-1} = RAR^{-1}$ is in the tangent space at the identity.

**Proposition:** For any $R \in G$, if $A$ is in the tangent space at the identity, $T_I G$, then $\underbrace{R A R^{-1}}_{\text{conjugation}}$, is also in $T_I G$.

**Proposition:** Let $R(t)$ be a curve in $G$ such that $R(0) = I$, \( R'(0) = B \). Let $A$ be an element of $T_I G$. Then:
\( R(t)AR(t)^{-1} \) is a curve in \( T_I G \)

\( \frac{d}{dt}|_{t=0} R(t)AR(t)^{-1} = BA - AB \)

**proof:** statement i) repeats the previous Proposition. To show statement ii) we must evaluate \( \frac{d}{dt}|_{t=0} [R(t)AR(t)^{-1}] \).

Note:

\[
R(t)R(t)^{-1} = I, \text{ hence } R(t)R(t)^{-1} + R(t) \left. \frac{d}{dt} (R(t)^{-1}) \right|_{t=0} \cdot 0 = B \cdot I + I \cdot \left. \frac{d}{dt} (R(t)^{-1}) \right|_{t=0} = BA - AB
\]

The expression \( BA - AB \) is known as the matrix Lie Bracket, \( [B, A] = BA - AB \).

**Proposition:** Given a matrix Lie Group \( G \), the tangent space at the identity \( T_I G \) is a Lie Algebra with respect to this Lie Bracket.

Note: Velocities “live” in some transformed space of Lie Algebras.

**Example 1:** If \( J \) is any nonsingular \( n \times n \) matrix, the set of all \( n \times n \) nonsingular matrices \( M \) such that \( M^TJM = J \) is a group (with respect to ordinary matrix multiplication).

**proof:** We will show that i) it is closed under matrix multiplication and ii) it is closed under the operation of taking inverses:

i)

\[
M_1^TJM_1 = J, \quad M_2^TJM_2 = J \quad \Rightarrow (M_1M_2)^TJM_1M_2 = M_2^T\overbrace{J}^{M_1^TJM_1}M_2 = J
\]

ii)

\[
M^TJM = J, \quad \text{does this imply } (M^{-1})^TJM^{-1} = J? \\
(M^TJM)M^{-1} = JM^{-1} \quad \Rightarrow M^TJ = JM^{-1} \quad \Rightarrow J = (M^T)^{-1}JM^{-1} = (M^{-1})^TJM^{-1}
\]

**SPECIAL CASE:** \( J = I, \quad \Rightarrow G = O(n), \quad n \times n \) orthogonal matrices.

**Example 2:** With \( n = 2m \) and

\[
J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix} \quad \Rightarrow G = Sp(2m), \quad \text{the symplectic group}
\]

**Example 3:**

\[
J = \begin{pmatrix} I_{n-1} & 0 \\ 0 & -1 \end{pmatrix} \quad \Rightarrow G \quad \text{the Lorentz group}
\]
Let us consider the tangent space at the identity $T_I G$. Let $R(t)$ be a curve in $G$. Then $R(t)^T J R(t) = J$ (The invariance property at the group level).

Assume $R(0) = I$, and write $R'(0) = A$. Differentiating both sides of the group invariance property at $t = 0$ we obtain:

$$R'(0)^T J + J R'(0) = A^T J + J A = 0$$

This is the corresponding invariance property for the Lie Algebra.

Special Case: $J = I$, $A^T + A = 0$.

Let $\mathcal{A}$ be a set of $n \times n$ matrices that is closed with respect to vector space operations and also with respect to the matrix Lie bracket $A, B, \in \mathcal{A} \implies [A, B] = AB - BA \in \mathcal{A}$. In other words $\mathcal{A}$ is a matrix Lie algebra. If $\mathcal{A}$ is such a Lie algebra, the set of all finite products:

$$e^{A_1} \cdot e^{A_2} \cdots e^{A_k}, \ k \in \mathbb{Z}^+, \ A_j \in \mathcal{A}, \ t_j \in \mathbb{R}$$

is the corresponding matrix Lie group.

**Example 1:** If $\mathcal{A}$ is the Lie algebra of all $n \times n$ matrices the corresponding Lie group is the group of $n \times n$ invertible matrices.

**Example 2:** Let $G = SO(3)$, (set of $3 \times 3$ orthogonal matrices with determinant equal to 1), and $\mathcal{A} = so(3)$ (set of $3 \times 3$ skew symmetric matrices).

Consider the basis for $so(3)$

$$\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{bmatrix}, \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{bmatrix}, \begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}$$

The Lie bracket of two of them gives the third (possible with a “-” sign) as:

$$\begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{bmatrix}$$

The group $SO(3)$ can be thought of as all products ($t_i s \in \mathbb{R}$):

$$e^{\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{bmatrix} t_1} e^{\begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} t_2} e^{\begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} t_3}$$

One can show that:

$$e^{\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{bmatrix} t_4} = \ldots$$
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix} t_1 + \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix} t_1^2 + \frac{1}{3!} \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix} t_1^3 + \frac{1}{4!} \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix} t_1^4 + \cdots
\]

\[
= \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos t_1 & -\sin t_1 \\
0 & \sin t_1 & \cos t_1
\end{pmatrix}
\]