Extended Jacobian

EXTENDED JACOBIAN APPROACH TO RESOLUTION OF REDUNDANCY:

If redundancy is resolved by means of a functional constraint $G(\theta) = 0$, we define the extended Jacobian:

$$J_e = \begin{pmatrix}
J \\
\frac{\partial G}{\partial \theta}
\end{pmatrix}$$

Before we proceed, let us return to the last example in Lecture 21 (i.e., the three link manipulator) and develop $J_e$ when $G(\theta) = \sin^2\theta_2 + \sin^2\theta_3$. The Extended Jacobian is given below, followed by two plots: the first is a 3-D plot of $\det J_e = 2(s_{23}(s_3c_3 - s_2c_2) + s_{23}(c_3 - c_2))$; and the second shows link angles $\theta_2, \theta_3$ that make $\det J_e = 0$.

$$J_e = \begin{pmatrix}
-s_1 - s_{12} - s_{123} \\
c_1 + c_{12} + c_{123} \\
0
\end{pmatrix}$$

Figure 1: 3-D Plot of $\det J_e$

Figure 2: $\theta_2, \theta_3$ that make $\det J_e$ equal to zero
Note that along trajectories satisfying the constraint that:

\[
\begin{pmatrix}
\dot{x} \\
0
\end{pmatrix} = J_e \dot{\theta}
\]

the joint space trajectories are formally given by the differential equation:

\[
\dot{\theta} = J_e^{-1} \begin{pmatrix}
\dot{x} \\
0
\end{pmatrix}
\]

But to make this rigorous we need to know when \( J_e \) will be singular. Certainly whenever \( J = \frac{\partial f}{\partial \theta} \) is singular, but by proper choice of \( G \), such mechanical singularities can be avoided. Are there other singularities?

\[
J_e = \begin{pmatrix}
J \\
G_\theta
\end{pmatrix}
\]

where the rows comprising \( G_\theta \) are partial derivatives of the constraint function. For simplicity we will restrict our discussion to the case \( m = n - 1 \), degree 1 redundancy.

Now

\[
J_e(J^\dagger : \vec{n}) = \begin{pmatrix}
J \\
G_\theta
\end{pmatrix} (J^T(JJ^T)^{-1} : \vec{n})
\]

\[
= \begin{pmatrix}
I \\
\xi_1 \cdots \xi_{n-1} & 0 & 0 & \cdots & 0
\end{pmatrix}
\]

(\(*\))

where \( J^\dagger = J^T(JJ^T)^{-1} \) and is an \( n \times (n-1) \) matrix. Thus there are potential problems in inverting \( J_e \Leftrightarrow G_\theta \cdot \vec{n} = 0 \Leftrightarrow (\text{one of } G_\theta \text{ or } \vec{n} \text{ are zero or } G_\theta \text{ lies in the row space of } J) \).

Note that (\(*\)) provides a formula whereby \( J_e^{-1} \) may be explicitly written as:

\[
J_e^{-1} = (J^\dagger : \vec{n}) \begin{pmatrix}
I \\
-\frac{\xi_1}{\xi_0} \xi^T \cdots -\frac{\xi_{n-1}}{\xi_0} \\
0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
J^\dagger - \vec{n} \cdot \xi^T \\
\frac{G_\theta \vec{n} \cdot \xi^T}{G_\theta \vec{n}}
\end{pmatrix}
\]

Now

\[
\dot{\theta} = J_e^{-1} \begin{pmatrix}
\dot{x} \\
0
\end{pmatrix} = J_e^\dagger \dot{x} - \frac{\vec{n} \cdot \xi^T}{G_\theta \vec{n}} \dot{x}
\]

and since

\[
\xi^T = (\xi_1, \ldots, \xi_{n-1}) = G_\theta J^\dagger
\]

we have

\[
\dot{\theta} = J_e^\dagger \dot{x} - \frac{\vec{n} \cdot G_\theta J^\dagger \dot{x}}{G_\theta \cdot \vec{n}} = J^\dagger \dot{x} + v(t) \vec{n}
\]
where

\[ v(t) = -\frac{G_\theta J^\dagger \dot{x}}{G_\theta \cdot \vec{n}} \]

Consider the conditions where invertibility of \( J_e \) is in question:

\[ G_\theta \cdot \vec{n} = 0 \]

Case: \( \vec{n} = 0 \)

\[ \vec{n} = \lambda(J_1, -J_2, \ldots, (-1)^{n+1}J_n) \]

where \( \lambda \) is a scalar and \( J_k \) is the \( k \)-th principal minor of \( J \)

\[ \vec{n} = 0 \iff \text{rank}J < m \iff \text{configuration is kinematically singular} \]

Case: \( G_\theta = 0 \)

In this case, numerator and denominator entries can cancel in

\[ v(t) = -\frac{G_\theta J^\dagger \dot{x}}{G_\theta \cdot \vec{n}} \]

resulting in the potential singularity being ignorable.

Case: \( \vec{n} \neq 0, \ G_\theta \neq 0 \)

This is a truly singular situation. We have \( G_\theta \) belonging to the row space of \( J \). The satisfaction of the constraint requires \( G_\theta \theta \equiv 0 \). But \( G_\theta = y^T J \) for some \( m \)-vector \( y \). Hence we must have

\[ y^T J \dot{\theta} = 0 \]

and thus \( y^T \dot{x} = 0 \) since \( \dot{x} = J \dot{\theta} \). Therefore, we cannot move the end-effector in the direction \( y \) and continue to satisfy the constraint.

Definition: Configurations corresponding to \( G_\theta \neq 0, \ \vec{n} \neq 0 \) but \( G_\theta \cdot \vec{n} = 0 \) are called algorithmic singularities.

Example

Consider the planar manipulator

\[
\begin{pmatrix}
    x \\
    y
\end{pmatrix} = 
\begin{pmatrix}
    c_1 + c_{12} + c_{123} \\
    s_1 + s_{12} + s_{123}
\end{pmatrix}
\]

The constraint maximizing \( \sin^2 \theta_2 + \sin^2 \theta_3 \) is \( \theta_2 = \theta_3 \). With this constraint, if we examine

\[
\frac{\partial G}{\partial \theta} \cdot J^\dagger
\]

\[
\frac{\partial G}{\partial \theta} \cdot \vec{n}
\]
we find that in lowest terms, the common denominator of the entries in this vector is:

\[(\sin \theta + \sin(2\theta))|\vec{n}|^2\]

This vanishes when $|\vec{n}|^2 = 0$ (not of interest) and when $\theta = \pm \frac{2\pi}{3}$. This is a true algorithmic singularity wherein the manipulator places its end-effector on its base.

For redundant manipulators:

1. Brockett showed that in general we cannot rule out the possibility of singularities.

2. While instantaneously optimizing a figure of merit regarding configurations provides a useful approach to resolution of kinematic redundancy, in general any such approach will be associated with algorithmic singularity.

Other work on the resolution of kinematic redundancy:

**Pathwise Resolution of Kinematic Redundancy**

Consider the following expression:

\[
\int_0^T \left( \frac{1}{2} \ddot{\theta}^T W^{-1} \dot{\theta} + g(\theta) \right) dt
\]

(A)

**Theorem** Joint space trajectories which optimize (A) satisfy

\[
\ddot{\theta} = J_W^\dagger (\ddot{x} - J \dot{\theta}) + P_W(WW^{-1} \dot{\theta} + W g_\theta)
\]

(\star\star)

where

\[
P_W = (I - J_W^\dagger J)
\]
is the weighted null space projection operator of the Jacobian, $J$, and $J_W^\dagger = WJ^T(JWJ^T)^{-1}$ and $J_W^\dagger$ is the weighted pseudo-inverse of $J$.

**Proof:** This theorem follows from a variational argument in which the joint space trajectory variations $\delta \theta$ are constrained to lie in the direction of the null space of $J$. The Euler-Lagrange operator defined by the functional (A) is:

$$\frac{d}{dt}(W^{-1}\dot{\theta}) - \frac{\partial g}{\partial \theta}$$

and if $\theta(\cdot)$ minimizes (A) over all trajectories which vary with respect to $\theta(\cdot)$ in the direction $\text{ker}J(\theta(t))$ (i.e., with respect to all trajectories corresponding to the given operational space path $x(\cdot)$), it follows that $\theta(\cdot)$ must satisfy

$$P_W\{W[\frac{d}{dt}(W^{-1}\dot{\theta}) - \frac{\partial g}{\partial \theta}]\} = 0 \quad (B)$$

Note that $P_W$ is the orthogonal projection onto the null space of $J$ to the inner product defined by the symmetric positive definite matrix $W^{-1}$. If we differentiate $\dot{x} = J\dot{\theta}$ with respect to $t$, we obtain

$$\ddot{x} = \dot{J}\dot{\theta} + J\ddot{\theta}$$

which may be equivalently written as

$$\ddot{\theta} = J_W^\dagger(\ddot{x} - \dot{J}\dot{\theta}) + P_W v \quad (C)$$

for an appropriate choice of $v$. Multiplying both sides of this equation by the projection operator $P_W$ and noting that $P_W J_W^\dagger = 0$, we obtain

$$P_W \ddot{\theta} = P_W v.$$ 

Using (B), we find that this implies

$$P_W v = \dot{W}W^{-1}\dot{\theta} + W\frac{\partial g}{\partial \theta}$$

and substituting this into (C) proves the theorem.

**Typical Boundary Conditions:**

I. Initial Value Conditions:

$$x(t_0) = f(\theta(t_0))$$

$$\dot{x}(t_0) = J(\theta(t_0))\dot{\theta}(t_0)$$

II. Two-point Boundary Values:

$$x(t_0) = f(\theta(t_0))$$

$$x(t_f) = f(\theta(t_f))$$
III. Natural Boundary Conditions:

\[ P_W \dot{\theta}(t_0) = 0 \]
\[ P_W \dot{\theta}(t_f) = 0 \]

IV. Periodic Boundary Conditions: if \( x \) satisfies \( x(0) = x(T) \), the objective is to find trajectories which satisfy (**) subject to \( \theta(0) = \theta(T) \) and \( \dot{\theta}(0) = \dot{\theta}(T) \)

**Homotopy Continuation Methods Applied to Path-wise Resolution of Kinematic Redundancy**

**Problem:**

\[
\min \int_0^T \left( \frac{\epsilon}{2} \| \dot{\theta} \|^2 + (1 - \epsilon) g(\theta) \right) dt
\]

with

\[
\begin{align*}
\theta(0) &= \theta(T), & \dot{\theta}(0) &= \dot{\theta}(T), \\
\epsilon \hat{n} \cdot \ddot{\theta} &= (1 - \epsilon) G(\theta)
\end{align*}
\]

where

\[ G(\theta) = \frac{\partial g}{\partial \theta}(\theta) \cdot \hat{n}(\theta) \]