Introduction to Lagrangian Mechanics

An idealized model of a robot has \( n \)-masses (point masses) interconnected by a set of links:

Let \((x_i, y_i, z_i)^T\) be the coordinates of the \( i \)-th point mass. Each

\[
\begin{pmatrix}
  x_i \\
  y_i \\
  z_i
\end{pmatrix} = \begin{pmatrix}
  x_i(\theta_1, \theta_2, \ldots, \theta_n) \\
  y_i(\theta_1, \theta_2, \ldots, \theta_n) \\
  z_i(\theta_1, \theta_2, \ldots, \theta_n)
\end{pmatrix}
\]

is a function of the joint angles \( \theta_i \). Let \( X_i, Y_i, Z_i \), be the components of the total force acting on the \( i \)-th point mass. Then we have:

\[
m_i \begin{pmatrix}
  \ddot{x}_i \\
  \ddot{y}_i \\
  \ddot{z}_i
\end{pmatrix} = \begin{pmatrix}
  X_i \\
  Y_i \\
  Z_i
\end{pmatrix}
\]

Multiplying both sides of these equations by:

\[
\begin{pmatrix}
  \frac{\partial x_i}{\partial \theta_j} \\
  \frac{\partial y_i}{\partial \theta_j} \\
  \frac{\partial z_i}{\partial \theta_j}
\end{pmatrix}
\]

we obtain (on summing):

Figure 1: Idealized Three Link Manipulator
\[
\sum_i m_i (\ddot{x}_i \frac{\partial x_i}{\partial \theta_j} + \ddot{y}_i \frac{\partial y_i}{\partial \theta_j} + \ddot{z}_i \frac{\partial z_i}{\partial \theta_j}) = \sum_i (X_i \frac{\partial x_i}{\partial \theta_j} + Y_i \frac{\partial y_i}{\partial \theta_j} + Z_i \frac{\partial z_i}{\partial \theta_j})
\]

But

\[
\frac{\partial \dot{x}_i}{\partial \theta_j} = \frac{\partial}{\partial \theta_j} (\frac{\partial x_i}{\partial \theta_1} + \frac{\partial x_i}{\partial \theta_2} + \ldots + \frac{\partial x_i}{\partial \theta_n}) = \frac{\partial x_i}{\partial \theta_j}
\]

Hence

\[
\dot{x}_i \frac{\partial x_i}{\partial \theta_j} = \dot{x}_i \frac{\partial x_i}{\partial \theta_j} = \frac{d}{dt} \left( \dot{x}_i \frac{\partial x_i}{\partial \theta_j} \right) = \dot{x}_i \frac{\partial}{\partial \theta_j} (\dot{x}_i) = \frac{d}{dt} \left( \dot{x}_i \frac{1}{2} \dot{x}_i^2 \right) - \frac{\partial}{\partial \theta_j} \left( \frac{1}{2} \dot{x}_i^2 \right)
\]

Returning to (\ref{eq:1}) we find that:

\[
\sum_i m_i (\ddot{x}_i \frac{\partial x_i}{\partial \theta_j} + \ddot{y}_i \frac{\partial y_i}{\partial \theta_j} + \ddot{z}_i \frac{\partial z_i}{\partial \theta_j}) = \sum_i \frac{d}{dt} \left( \frac{\partial}{\partial \theta_j} \left( \frac{1}{2} m_i \dot{x}_i^2 \right) - \frac{\partial}{\partial \theta_j} \left( \frac{1}{2} m_i \dot{y}_i^2 \right) \right)
\]

(*)

\[
= \frac{d}{dt} \frac{\partial}{\partial \theta_j} \left( \sum_i \frac{1}{2} m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) \right) - \frac{\partial}{\partial \theta_j} \left( \sum_i \frac{1}{2} m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) \right)
\]

(***)

We next observe that all terms in the expression are functions of \(\theta_1, \theta_2, \ldots, \theta_n, \dot{\theta}_1, \dot{\theta}_2, \ldots, \dot{\theta}_n\). Also \(x_i = x_i(\theta_1, \ldots, \theta_n)\). Hence:

\[
\dot{x}_i = \frac{d}{dt} (x_i(\theta_1, \ldots, \theta_n)) = \frac{\partial x_i}{\partial \theta_1} \dot{\theta}_1 + \ldots + \frac{\partial x_i}{\partial \theta_n} \dot{\theta}_n
\]
Similarly,

\[
\dot{y}_i = \frac{\partial y_i}{\partial \theta_1} \dot{\theta}_1 + \ldots + \frac{\partial y_i}{\partial \theta_n} \dot{\theta}_n
\]

\[
\dot{z}_i = \frac{\partial z_i}{\partial \theta_1} \dot{\theta}_1 + \ldots + \frac{\partial z_i}{\partial \theta_n} \dot{\theta}_n
\]

Hence,

\[
\sum_{i=1}^{n} \frac{1}{2} m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) = \text{kinetic energy of the system} = T(\theta_1, \theta_2, \ldots, \theta_n, \dot{\theta}_1, \dot{\theta}_2, \ldots, \dot{\theta}_n)
\]

The expression (\(\star\)) is just

\[
\frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}_j} - \frac{\partial T}{\partial \theta_j}
\]

and equation (\(\star\)) becomes:

\[
\star' \quad \frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}_j} - \frac{\partial T}{\partial \theta_j} = \sum_{i=1}^{n} \left( \frac{\partial x_i}{\partial \theta_j} \frac{\partial y_i}{\partial \theta_j} \frac{\partial z_i}{\partial \theta_j} \right) \begin{pmatrix} X_i \\ Y_i \\ Z_i \end{pmatrix}
\]

This expression will be distilled still further but to do this we need an amazing fact about transmission of forces through a linkage. Consider the position relationship:

\[
\vec{x}_i = f_i(\theta_1, \ldots, \theta_n)
\]

(for the i-th particle in our system). Suppose there are no forces acting on the system except a force \(\vec{F}_i\) on \(\vec{x}_i\). Let this force move the system by an amount \(\delta \vec{x}_i\) corresponding to (infinitesimal) movement \(\delta \theta_j\), for \((j = 1, \ldots, n)\) of each joint. Then with \(\vec{\tau}_j\) being the torques induced in the joints we have:

\[
\delta \vec{x}_i \cdot \vec{F}_i = \sum_j \delta \theta_j \vec{\tau}_j
\]

(work measured w.r.t the i-th particle)

\[
\text{total work from joint motions (where } \vec{\tau}_j \text{ are torques induced in the joints)}
\]

This equation may be re-written as:

\[
\delta \vec{x}_i \cdot \vec{F}_i = \vec{\tau} \cdot \delta \vec{\theta}
\]

Now
\[ \delta \vec{x}_i = \frac{\partial f_i}{\partial \theta} \cdot \delta \vec{\theta} \]

\[ \Rightarrow \]

\[ \delta \vec{x}_i \cdot \vec{F}_i = \delta \vec{\theta}^T (\frac{\partial f_i}{\partial \theta})^T \cdot \vec{F}_i \]

Hence, we have:

\[ \delta \vec{\theta}^T (\frac{\partial f_i}{\partial \theta})^T \cdot \vec{F}_i = \delta \vec{\theta}^T \vec{\tau} \]

This equation is valid for any “imaginary” infinitesimal displacement \( \delta \vec{\theta} \) (PRINCIPLE OF VIRTUAL WORK).

Hence:

\[ (\frac{\partial f_i}{\partial \theta})^T \cdot \vec{F}_i = \vec{\tau} \]

Consider now the manipulator shown below where \( x = f(\theta) \) and the Jacobian is given by \( J = \frac{\partial f}{\partial \theta} \). What the manipulator Jacobian tells us is the following:

\[ \dot{x} = J \dot{\theta} \]

how velocities are transmitted from joints to the end-effector, and

\[ \vec{\tau} = J^T \vec{F} \]

how a force vector at the end-effector is felt as torques at the joints.

Figure 2: Manipulator with End-Effector

What about singularities in \( J^T \)? For effective transmission of forces, singularities are a good thing.

From above we have:
\[
m \left( \begin{array}{c}
 \ddot{x}_i \\
 \ddot{y}_i \\
 \ddot{z}_i
\end{array} \right) = \left( \begin{array}{c}
 X_i \\
 Y_i \\
 Z_i
\end{array} \right)
\]

\[
\sum_i m_i (\ddot{x}_i + \dot{y}_i \frac{\partial y_i}{\partial \theta_j} + \ddot{z}_i \frac{\partial z_i}{\partial \theta_j}) = \sum_i (X_i \frac{\partial x_i}{\partial \theta_j} + Y_i \frac{\partial y_i}{\partial \theta_j} + Z_i \frac{\partial z_i}{\partial \theta_j})
\]

\[
\frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}_j} - \frac{\partial T}{\partial \theta_j} = \sum_i (X_i \frac{\partial x_i}{\partial \theta_j} + Y_i \frac{\partial y_i}{\partial \theta_j} + Z_i \frac{\partial z_i}{\partial \theta_j})
\]

and

\[
\dot{x}_i = \frac{\partial x_i}{\partial \theta_1} \dot{\theta}_1 + \ldots + \frac{\partial x_i}{\partial \theta_n} \dot{\theta}_n
\]

where the kinetic energy is given by:

\[
T = \sum_i \frac{1}{2} m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)
\]

This can be expressed as:

\[
T(\theta_1, \theta_2, \ldots, \theta_n) = \frac{1}{2} (\dot{\theta}_1, \dot{\theta}_2, \ldots, \dot{\theta}_n)^T M(\theta_1, \theta_2, \ldots, \theta_n) \left( \begin{array}{c}
 \dot{\theta}_1 \\
 \dot{\theta}_2 \\
 \vdots \\
 \dot{\theta}_n
\end{array} \right)
\]

where \(M(\theta_1, \theta_2, \ldots, \theta_n)\) is a symmetric, positive semidefinite (the components of which are left to you to work out as an exercise).

In terms of the notation for the problem at hand we have:

\[
f_i(\theta) = \left( \begin{array}{c}
 x_i(\theta) \\
 y_i(\theta) \\
 z_i(\theta)
\end{array} \right) = \left( \begin{array}{c}
 x_i(\theta_1, \theta_2, \ldots, \theta_n) \\
 y_i(\theta_1, \theta_2, \ldots, \theta_n) \\
 z_i(\theta_1, \theta_2, \ldots, \theta_n)
\end{array} \right)
\]

\[
\frac{\partial f_i}{\partial \theta} = \left( \begin{array}{ccc}
 \frac{\partial x_i}{\partial \theta_1} & \frac{\partial x_i}{\partial \theta_2} & \ldots & \frac{\partial x_i}{\partial \theta_n} \\
 \frac{\partial y_i}{\partial \theta_1} & \frac{\partial y_i}{\partial \theta_2} & \ldots & \frac{\partial y_i}{\partial \theta_n} \\
 \frac{\partial z_i}{\partial \theta_1} & \frac{\partial z_i}{\partial \theta_2} & \ldots & \frac{\partial z_i}{\partial \theta_n}
\end{array} \right)
\]

The force relation is:

\[
\tau = \left( \begin{array}{c}
 \tau_1 \\
 \tau_2 \\
 \vdots \\
 \tau_n
\end{array} \right) = \left( \begin{array}{ccc}
 \frac{\partial x_i}{\partial \theta_1} & \frac{\partial y_i}{\partial \theta_1} & \frac{\partial z_i}{\partial \theta_1} \\
 \frac{\partial x_i}{\partial \theta_2} & \frac{\partial y_i}{\partial \theta_2} & \frac{\partial z_i}{\partial \theta_2} \\
 \vdots & \vdots & \vdots \\
 \frac{\partial x_i}{\partial \theta_n} & \frac{\partial y_i}{\partial \theta_n} & \frac{\partial z_i}{\partial \theta_n}
\end{array} \right) \left( \begin{array}{c}
 X_i \\
 Y_i \\
 Z_i
\end{array} \right)
\]
(Actually, there should be a superscript or something on the $\tau_i$'s to record the fact that these are torques corresponding to the force acting on the i-th point mass.)

Returning to $\star'\,$ we have:

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}_j} - \frac{\partial T}{\partial \theta_j} = \sum_{i=1}^{n} \left( \frac{\partial x_i}{\partial \theta_j} \frac{\partial y_i}{\partial \theta_j} \frac{\partial z_i}{\partial \theta_j} \right) \left( \begin{array}{c} X_i \\ Y_i \\ Z_i \end{array} \right) = \sum_{i=1}^{n} \tau^i_j = Q_j$$

where $\tau^i_j$ is the force/torque at the j-th joint corresponding to the net force applied to the i-th point mass, and $Q_j$ is the net force/torque on the j-th joint.

**Theorem 1:** Suppose the configuration of a dynamical system may be specified by coordinates $\theta_1, \theta_2, \ldots, \theta_n$, and suppose the kinetic energy (corresponding to any possible motion) may be written $T = T(\theta_1, \theta_2, \ldots, \theta_n; \dot{\theta}_1, \dot{\theta}_2, \ldots, \dot{\theta}_n)$, then the equations of motion for this system are given by:

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}_i} - \frac{\partial T}{\partial \theta_i} = Q_i \quad (i = 1, \ldots, n)$$

where $Q_i$ is the i-th generalized force acting on the system.

**Remark 1:** In robotics, this formulation of Newton’s 2nd Law is quite natural, since torques and forces from actuators are applied to the joints.

**Remark 2:** There may be a potential energy function $V(\theta_1, \theta_2, \ldots, \theta_n)$ denoting the path independent work done in moving the system from some reference configuration $(\theta^*_1, \theta^*_2, \ldots, \theta^*_n)$ to $(\theta_1, \theta_2, \ldots, \theta_n)$. This potential energy gives rise to conservative generalized forces $Q_j = -\frac{\partial V}{\partial \theta_j}$.

**Theorem 2:** Suppose the configuration of a dynamical system may be specified by coordinates $\theta_1, \theta_2, \ldots, \theta_n$, and suppose kinetic energy (corresponding to any possible motion) may be written $T = T(\theta_1, \theta_2, \ldots, \theta_n; \dot{\theta}_1, \dot{\theta}_2, \ldots, \dot{\theta}_n)$, and the potential energy (corresponding to any possible configuration) may be written $V = V(\theta_1, \theta_2, \ldots, \theta_n)$, then the equations of motion for this system are given by:

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}_i} - \frac{\partial T}{\partial \theta_i} = -\frac{\partial V}{\partial \theta_i} + \tau_i \quad (i = 1, \ldots, n)$$

where $\tau_i$ is the i-th generalized (exogenous) force affecting the variable $\theta_i$.

**Corollary:** If we define the Lagrangian $L = T - V$, then Lagrange’s equations of motion (or the Euler-Lagrange equations) may be written

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} = \tau_i \quad (i = 1, \ldots, n)$$

**Hamilton’s Interpretation: Principle of Least Actions**

Define the action:

$$S = \int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1} T - V dt$$
Then $\delta S = 0$, where $\delta S$ = variation in $S$ that occurs when we vary the coordinates of the path $(\theta, \dot{\theta})$ by an infinitesimal amount. When variation about given path is thus $= 0$, the value of $S$ along the path may be a local maximum or minimum in the set of all paths having the same beginning and end points. This is like a first derivative test.

$$S + \delta S = \int_{t_0}^{t_1} L(\theta + \delta \theta, \dot{\theta} + \delta \dot{\theta}) dt$$

$$= \int_{t_0}^{t_1} L(\theta, \dot{\theta}) + \frac{\partial L}{\partial \theta} \delta \theta + \frac{\partial L}{\partial \dot{\theta}} \delta \dot{\theta} + o(\delta \theta) \, dt$$

$$= \int_{t_0}^{t_1} L(\theta, \dot{\theta}) + \frac{\partial L}{\partial \theta} \delta \theta - \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} \delta \theta + o(\delta \theta) dt$$

This last equation comes from the fact that:

$$\int_{t_0}^{t_1} \frac{\partial L}{\partial \dot{\theta}} \delta \dot{\theta} dt = \left. \frac{\partial L}{\partial \theta} \delta \theta \right|_{t_0}^{t_1} - \int_{t_0}^{t_1} \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} \delta \theta dt$$

where that first term on the right is zero because $\theta(t)$ and $\theta(t) + \delta \theta(t)$ have the same endpoints at $t_0, t_1$.

Now suppose $(\theta(t), \dot{\theta}(t))$ defines a curve minimizing $S$ among all possible curves taking on prescribed end point values at $t_0, t_1$. Then along this curve:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

Otherwise, let

$$\delta \theta(t) = \epsilon \left( \frac{d}{dt} \frac{\partial L}{\partial \theta} - \frac{\partial L}{\partial \theta} \right) \quad (\epsilon > 0)$$

then

$$S + \delta S = \int_{t_0}^{t_1} L(\theta, \dot{\theta}) dt - \int_{t_0}^{t_1} \epsilon \left( \frac{d}{dt} \frac{\partial L}{\partial \theta} - \frac{\partial L}{\partial \theta} \right)^2 dt + o(\epsilon)$$

For $\epsilon > 0$ sufficiently small, $S + \delta S < S$, contradicting the assumed local minimality of $S$.

Therefore, trajectories which minimize the action $\int_{t_0}^{t_1} L(\theta, \dot{\theta}) dt$ satisfy the Euler-Lagrange equation

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$