Solutions

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ENG ME/SE 740:

Exercises (Set 3)  (Due 3/5/19)

2 pts.
1. Prove that for any square matrices $A$ and $B$ that $e^{(A+B)t} = e^{At}e^{Bt} \iff [A, B] = 0$.

2 pts 2. For the following $2 \times 2$ matrices, compute $e^{At}$:

(a) $A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$
(b) $A = \begin{pmatrix} -a & a \\ b & -b \end{pmatrix}$

6 pts 3. Find $\theta$, $\omega$ and $v$ such that

$$\text{Exp} \left[ \begin{pmatrix} \hat{\omega} \\ 0 \\ 0 \end{pmatrix} \theta \right] = \begin{pmatrix} \frac{1+\sqrt{2}}{3} & \frac{2-\sqrt{2}(1+\sqrt{3})}{6} & \frac{2+\sqrt{2}(-1+\sqrt{3})}{6} & \frac{\sqrt{2}}{3} + \frac{\pi}{12} \\ \frac{2+\sqrt{2}(1+\sqrt{3})}{6} & \frac{1+\sqrt{2}}{3} & \frac{2-\sqrt{2}(1+\sqrt{3})}{6} & -\frac{1}{3\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}} + \frac{\pi}{12} \\ \frac{2-\sqrt{2}(1+\sqrt{3})}{6} & \frac{1+\sqrt{2}}{3} & \frac{2+\sqrt{2}(1+\sqrt{3})}{6} & -\frac{1}{3\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{6}} + \frac{\pi}{12} \\ \frac{2+\sqrt{2}(1+\sqrt{3})}{6} & \frac{1+\sqrt{2}}{3} & \frac{2-\sqrt{2}(1+\sqrt{3})}{6} & -\frac{1}{3\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}} + \frac{\pi}{12} \end{pmatrix}$$
1. 

\[ e^{(A+B)t} = I + (A+B)t + \frac{1}{2}(A+B)^2t^2 + \frac{1}{3!}(A+B)^3t^3 + \ldots \]

\[ = I + (A+B)t + \frac{1}{2}(A+AB+BA+A^2)t^2 \]

\[ + \frac{1}{3!} \left( A^3 + A^2B + ABA + BA^2 + AB^2 + BAB + B^2A + B^3 \right)t^3 \]

\[ + \ldots \]

\[ e^{At}e^{Bt} = (I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \ldots) \]

\[ \cdot (I + Bt + \frac{1}{2}B^2t^2 + \frac{1}{3!}B^3t^3 + \ldots) \]

\[ = I + At + Bt + \frac{1}{2}A^2t^2 + ABt^2 + \frac{1}{2}B^2t^2 \]

\[ + \frac{1}{3!} A^3t^3 + \frac{1}{2}A^2Bt^2 + \frac{1}{2}AB^2t^2 + \frac{1}{3!}B^3t^3 \]

The terms of all orders \(2\) and higher in \(e^{(A+B)t}\) will have factors \(A, B\) appearing in all possible orders. The corresponding terms in \(e^{At}e^{Bt}\) will be of the form \(A^k B^{n-k}\). These can be equated \(\iff\) \(AB = BA\).

2(a) 

\[ e^{(A-B)t} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} e^{xt} & -e^{xt} \\ e^{xt} & e^{xt} \end{pmatrix} \]

\[ = \begin{pmatrix} e^{xt} \cos t & -e^{xt} \sin t \\ e^{xt} \sin t & e^{xt} \cos t \end{pmatrix} \]
2(b) \[ e^{(-a \ b-b) t} \]

\[ = I - \frac{A}{a+b} \left( 1 - \frac{(a+b)t}{2} + \frac{(a+b)^2 t^2}{3!} - \frac{(a+b)^3 t^3}{4!} + \ldots \right) + \frac{A}{a+b} \]

where \[ \Lambda = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \]

\[ = \left(1 + \frac{a}{a+b} \right) \begin{pmatrix} a & b \\ b & a \end{pmatrix} \left(1 - \frac{b}{a+b} \right) - \frac{1}{a+b} \begin{pmatrix} -a & a \\ b & -b \end{pmatrix} e^{-(a+b)t} \]

\[ = \begin{pmatrix} \frac{b}{a+b} + \frac{a}{a+b} e^{-(a+b)t} & \frac{a}{a+b} - \frac{a}{a+b} e^{-(a+b)t} \\ \frac{b}{a+b} - \frac{b}{a+b} e^{-(a+b)t} & \frac{a}{a+b} + \frac{b}{a+b} e^{-(a+b)t} \end{pmatrix} \]
\[ \exp\left( \begin{bmatrix} \frac{1+\sqrt{2}}{3} & \frac{2-\sqrt{2}(1+\sqrt{3})}{6} & \frac{2+\sqrt{2}(-1+\sqrt{3})}{6} & \frac{\sqrt{2}}{3} + \frac{\pi}{12} \\ \frac{2+\sqrt{2}(-1+\sqrt{3})}{6} & \frac{1+\sqrt{2}}{3} & \frac{2-\sqrt{2}(1+\sqrt{3})}{6} & -\frac{1}{3\sqrt{2}} + \frac{\sqrt{3}}{6} + \frac{\pi}{12} \\ \frac{2-\sqrt{2}(1+\sqrt{3})}{6} & \frac{2+\sqrt{2}(-1+\sqrt{3})}{6} & \frac{1+\sqrt{2}}{3} & -\frac{1}{3\sqrt{2}} - \frac{\sqrt{3}}{6} + \frac{\pi}{12} \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \]

\[ \exp\left( \begin{bmatrix} \hat{\omega} \\ 0 \end{bmatrix} \right) = e^{\hat{\omega} \cdot \theta} \left[ (I - R) \cdot \hat{\omega} + (\hat{\omega}^2 + 1) \cdot \theta \right] \cdot v \]

\[ R = e^{\hat{\omega} \cdot \theta} = \begin{bmatrix}
-\omega_2 \cdot \sin(\theta) + \omega_1 \cdot \omega_3 \cdot (1 - \cos(\theta)) & \omega_3 \cdot \sin(\theta) + \omega_1 \cdot \omega_2 \cdot (1 - \cos(\theta)) & 1 - \left( \omega_1^2 + \omega_2^2 \right) \cdot (1 - \cos(\theta)) \\
\omega_2 \cdot \sin(\theta) + \omega_1 \cdot \omega_2 \cdot (1 - \cos(\theta)) & -\omega_2 \cdot \sin(\theta) + \omega_1 \cdot \omega_3 \cdot (1 - \cos(\theta)) & \omega_3 \cdot \sin(\theta) + \omega_1 \cdot \omega_2 \cdot (1 - \cos(\theta)) \\
\omega_3 \cdot \sin(\theta) + \omega_1 \cdot \omega_2 \cdot (1 - \cos(\theta)) & \omega_3 \cdot \sin(\theta) + \omega_1 \cdot \omega_2 \cdot (1 - \cos(\theta)) & 1 - \left( \omega_1^2 + \omega_2^2 \right) \cdot (1 - \cos(\theta))
\end{bmatrix}
\]

\[ R := \begin{bmatrix}
\frac{1+\sqrt{2}}{3} & \frac{2-\sqrt{2}(1+\sqrt{3})}{6} & \frac{2+\sqrt{2}(-1+\sqrt{3})}{6} \\
\frac{2+\sqrt{2}(-1+\sqrt{3})}{6} & \frac{1+\sqrt{2}}{3} & \frac{2-\sqrt{2}(1+\sqrt{3})}{6} \\
\frac{2-\sqrt{2}(1+\sqrt{3})}{6} & \frac{2+\sqrt{2}(-1+\sqrt{3})}{6} & \frac{1+\sqrt{2}}{3}
\end{bmatrix} = \begin{bmatrix}
0.805 & -0.311 & 0.506 \\
0.506 & 0.805 & -0.311 \\
-0.311 & 0.506 & 0.805
\end{bmatrix}
\]

\[ \theta := \cos\left( \frac{\text{tr}(R) - 1}{2} \right) = 0.785 = \frac{\pi}{4} \]

\[ \omega_1 := \frac{(R_{3,2} - R_{2,3})}{2 \cdot \sin(\theta)} \rightarrow 0.577 \quad \omega_2 := \frac{R_{1,3} - R_{3,1}}{2 \cdot \sin(\theta)} \rightarrow 0.577 \quad \omega_3 := \frac{R_{2,1} - R_{1,2}}{2 \cdot \sin(\theta)} \rightarrow 0.577 \]

\[ \hat{\omega} := \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -0.577 & 0.577 \\ 0.577 & 0 & -0.577 \\ -0.577 & 0.577 & 0 \end{bmatrix} \quad I := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

Given

\[ \begin{bmatrix}
\frac{\sqrt{2}}{3} + \frac{\pi}{12} \\
-\frac{1}{3\sqrt{2}} + \frac{\sqrt{3}}{6} + \frac{\pi}{12} \\
-\frac{1}{3\sqrt{2}} - \frac{\sqrt{3}}{6} + \frac{\pi}{12}
\end{bmatrix} = \left[ (I - R) \cdot \hat{\omega} + (\hat{\omega}^2 + 1) \cdot \theta \right] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \]

\[ \text{find}(v_1, v_2, v_3) \rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \]