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Solutions

ENG ME 740:

Exercises (Set 5)

3pts. 1. Consider the kinematics function associated with a *spherical wrist* mechanism.

- correct partials
- correct 3x3 Jacobian
- correct explanation of singularities

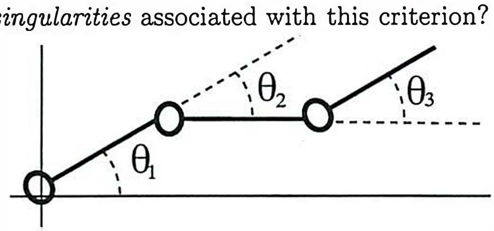
$$\begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\end{pmatrix} \mapsto \begin{pmatrix}\cos\theta_1 & -\sin\theta_1 & 0\\ \sin\theta_1 & \cos\theta_1 & 0\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}\cos\theta_2 & -\sin\theta_2 & 0\\ \sin\theta_2 & \cos\theta_2 & 0\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}$$
$$\cdot\begin{pmatrix}\cos\theta_3 & -\sin\theta_3 & 0\\ \sin\theta_3 & \cos\theta_3 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Compute the manipulator Jacobian (in the form of a 3×3 matrix) for this function. Sketch the mechanism, describe kinematically singular configurations in terms of your sketch, and explain the physical significance of these singularities.

2. A planar three link mechanism (See figure below.) is viewed as a (positionally) redundant manipulator. Assume the links all have equal length, and we assign (relative angle) coordinates θ_1, θ_2, and θ_3 in the usual way. Consider configurations which place the end-effector at prescribed (x, y)-coordinates so as to maximize the criterion $g(\theta) = \sin^2\theta_2 + \sin^2\theta_3$. The necessary condition for this placement to be optimal (with respect to maximizing g) is $\frac{\partial g}{\partial \theta}\cdot\vec{n} = 0$ where $\vec{n}$ is the (unit) null-space vector of the manipulator Jacobian.

2pts. (a) What configurations satisfy this necessary condition? — something beyond what we did in class

1pt. (b) What are the *algorithmic singularities* associated with this criterion?

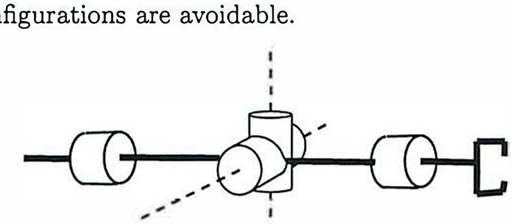


3. The mechanism in the figure has four successive joints, with each joint axis $\perp$ to its predecessor and successor as shown.

1pt. (a) Write down the mechanism Jacobian as a 3×4 matrix (recall this from class).

2pts. (b) Sketch the kinematically singular configurations, and explain how velocities are physically restricted when the mechanism is in such a configuration.

1pt. (c) Explain why all singular configurations are avoidable.



ME 740 EXERCISES **Set 5**

SOLUTIONS

PROF. BAILLIEUL

1. $$T(\theta_1,\theta_2,\theta_3) = \begin{pmatrix} c\theta_1 & -s\theta_1 & 0 \\ s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} c\theta_2 & -s\theta_2 & 0 \\ s\theta_2 & c\theta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} c\theta_3 & -s\theta_3 & 0 \\ s\theta_3 & c\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\frac{\partial T}{\partial \theta_1} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} T$$

$$\frac{\partial T}{\partial \theta_2} = \begin{pmatrix} c\theta_1 & -s\theta_1 & 0 \\ s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} c\theta_1 & s\theta_1 & 0 \\ -s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} T$$

$$= \begin{pmatrix} 0 & 0 & -c\theta_1 \\ 0 & 0 & -s\theta_1 \\ c\theta_1 & s\theta_1 & 0 \end{pmatrix} T$$

$$\frac{\partial T}{\partial \theta_3} = \begin{pmatrix} 0 & c\theta_2 & s\theta_1 s\theta_2 \\ -c\theta_2 & 0 & -c\theta_1 s\theta_2 \\ -s\theta_1 s\theta_2 & c\theta_1 s\theta_2 & 0 \end{pmatrix} T$$

1 cont.)

$$\begin{pmatrix} 0 & -d\omega_z & d\omega_y \\ d\omega_z & 0 & -d\omega_x \\ -d\omega_y & d\omega_x & 0 \end{pmatrix} T = \sum_{k=1}^{3} \frac{\partial T}{\partial \theta_k} d\theta_k .$$

$$\Downarrow$$

$$\begin{pmatrix} d\omega_x \\ d\omega_y \\ d\omega_z \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & s\theta_1 & c\theta_1 s\theta_2 \\ 0 & -c\theta_1 & s\theta_1 s\theta_2 \\ 1 & 0 & -c\theta_2 \end{pmatrix}}_{J} \begin{pmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \end{pmatrix}$$

$$\det J = \sin\theta_2$$

$\theta_2 = 0, \pi$ defines the singular configurations.

In the singular configurations the axes z_1, z_3 are aligned. In these configurations, no instantaneous velocity component about $z_1 \times z_2$ is possible.

2.(a)
$$J = \begin{pmatrix} -S_1 - S_{12} - S_{123} & -S_{12} - S_{123} & -S_{123} \\ C_1 + C_{12} + C_{123} & C_{12} + C_{123} & C_{123} \end{pmatrix}$$

One way to find the null-space of J is to take the cross product of the rows:

$$n_J = \begin{pmatrix} \sin\theta_3 \\ -\sin\theta_3 - \sin(\theta_2+\theta_3) \\ \sin\theta_2 + \sin(\theta_2+\theta_3) \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial g}{\partial \theta_1} \\ \frac{\partial g}{\partial \theta_2} \\ \frac{\partial g}{\partial \theta_3} \end{pmatrix} = \begin{pmatrix} 0 \\ 2 S_2 C_2 \\ 2 S_3 C_3 \end{pmatrix}$$

$$\frac{\partial g}{\partial \theta} \cdot n_J = 2\left[-S_2C_2(S_3+S_{23}) + S_3C_3(S_2+S_{23}) \right]$$

Among the solutions to $\frac{\partial g}{\partial \theta} \cdot n_J = 0$ is one that specifies $\theta_2 = \theta_3$. This solution is preferrable to, say $\theta_3 = -\theta_2$, since it avoids the singularities of J.

OTHER SOLUTIONS TO $\frac{\partial g}{\partial \theta} \cdot n_J = 0$ SUCH AS

$\theta_2 = \frac{2m+1}{2}\pi$, $m = 0, \pm 1, \ldots$ $\theta_3 = n\pi$, $n = 0, \pm 1$

OR $\theta_2 = m\pi$, $\theta_3 = \frac{(2n+1)\pi}{2}$

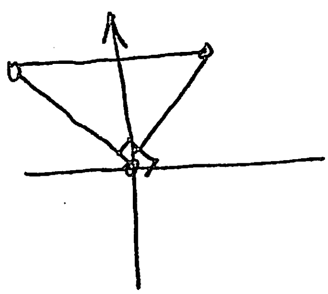
DO NOT MAXIMIZE $\sin^2\theta_2 + \sin^2\theta_3$.

2(b) $G(\theta_1, \theta_2\, \theta_3) = \theta_2 - \theta_3$

$$\frac{\partial G}{\partial \theta} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \qquad \frac{\partial G}{\partial \theta} \cdot \eta_J = -2\left(S_2 + S_{22}\right)$$

$$= -2\left(\sin\theta_2 + \sin 2\theta_2\right)$$

$\sin\theta_2 + \sin 2\theta_2 = 0 \iff \theta_2 = \pm 2\pi/3 \pmod{2\pi}$. [There is also $\theta_2 = 0, \pi$, but we rule these out - sing. of

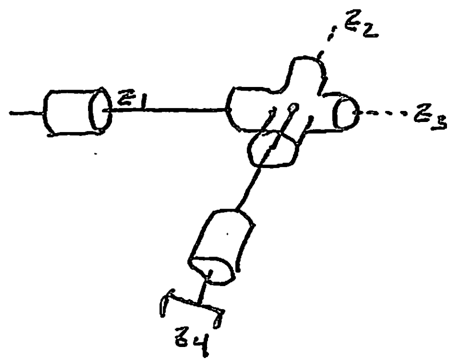


When $\theta_2 = \theta_3$ and $\theta_2 = \pm 2\pi/3$, the manipulator has its tip on its base.

A NOTE REGARDING SOLUTIONS $\theta_3 = -\theta_2$: SUCH SOLUTIONS ARE ALSO LOCALLY MAXIMIZING AND WHEN THE MECHANISM IS NEAR TO ITS FULL EXTENSION, THE METRIC $\sin^2\theta_2 + \sin^2\theta_3$ ASSUMES VALUE LARGER THAN FOR THE $\theta_2 = \theta_3$ SOLUTION.

3.(a)
$$\begin{pmatrix} 0 & S_1 & S_2C_1 & -S_1C_3 + C_1C_2S_3 \\ 0 & -C_1 & S_2S_1 & C_1C_3 + S_1C_2S_3 \\ 1 & 0 & -C_2 & S_2S_3 \end{pmatrix} = \begin{pmatrix} C_1 & -S_1 & 0 \\ S_1 & C_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & S_2 & C_2S_3 \\ 0 & -1 & 0 & C_3 \\ 1 & 0 & -C_2 & S_2S_3 \end{pmatrix}$$

(b)



The kinematically singular configurati have $\theta_2 = 0, \pi$, $\theta_3 = 0, \pi$. In these cases we have $z_4 \parallel z_2$ and $z_1 \parallel z_3$. Ang. vel. of the end-effector is (instantaneously) restricted to a plan

(c) In all case, we can orient the hand as it would be in a singular configuration without J losing rank.