

ME/SE 740

Lecture 20

Infinitesimal Motions

Recall Euler's Theorem (Lecture 9) where we showed that every 3×3 proper rotation $X \in SO(3)$ is a rotation about an axis \vec{k} by a certain amount θ . We had developed the following expression:

$$R(\theta, \vec{k}) = \begin{pmatrix} k_x^2(1 - \cos \theta) + \cos \theta & -k_z \sin \theta + k_x k_y(1 - \cos \theta) & k_y \sin \theta + k_x k_z(1 - \cos \theta) \\ k_z \sin \theta + k_x k_y(1 - \cos \theta) & k_y^2(1 - \cos \theta) + \cos \theta & -k_x \sin \theta + k_y k_z(1 - \cos \theta) \\ -k_y \sin \theta + k_x k_z(1 - \cos \theta) & k_x \sin \theta + k_y k_z(1 - \cos \theta) & k_z^2(1 - \cos \theta) + \cos \theta \end{pmatrix}$$

Replacing $\theta \rightarrow d\theta$ in the above expression generates:

$$\begin{pmatrix} 1 & -k_z d\theta & k_y d\theta \\ k_z d\theta & 1 & -k_x d\theta \\ -k_y d\theta & k_x d\theta & 1 \end{pmatrix} \implies dR(\theta, \vec{k}) = \begin{pmatrix} 0 & -k_z d\theta & k_y d\theta \\ k_z d\theta & 0 & -k_x d\theta \\ -k_y d\theta & k_x d\theta & 0 \end{pmatrix}$$

Then

$$\begin{aligned} R(x, \delta_x)R(y, \delta_y)R(z, \delta_z) &\sim R(y, \delta_y)R(z, \delta_z)R(x, \delta_x) \sim \text{in any order} \\ &\sim \begin{pmatrix} 1 & -\delta_z & \delta_y \\ \delta_z & 1 & -\delta_x \\ -\delta_y & \delta_x & 1 \end{pmatrix} \end{aligned}$$

So we can write:

$$dR = \begin{pmatrix} 0 & -\delta_z & \delta_y \\ \delta_z & 0 & -\delta_x \\ -\delta_y & \delta_x & 0 \end{pmatrix}$$

Putting in translations:

$$\begin{aligned} dT &= \underbrace{\begin{pmatrix} 0 & -\delta_z & \delta_y & dx \\ \delta_z & 0 & -\delta_x & dy \\ -\delta_y & \delta_x & 0 & dz \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{\Delta} T \\ &= T \underbrace{\begin{pmatrix} 0 & -\delta_z^T & \delta_y^T & dx^T \\ \delta_z^T & 0 & -\delta_x^T & dy^T \\ -\delta_y^T & \delta_x^T & 0 & dz \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{\Delta^T} \end{aligned}$$

where Δ, Δ^T are infinitesimal motions and T microscopic, and Δ is in base frame whereas Δ^T is in tool frame.

Suppose that $T = AB$:

$$dT = \Delta T = \Delta AB = A\Delta^A B = A\Delta^A A^{-1} AB = A\Delta^A A^{-1} T$$

This implies that: $\Delta = A\Delta^A A^{-1}$. We can also write: $dT = T\Delta^T = AB\Delta^T = A\Delta^A B = ABB^{-1}\Delta^A B = TB^{-1}\Delta^A B$ which implies that $\Delta^T = B^{-1}\Delta^A B$.

This circle of ideas is key to computing the Jacobian of Kinematics transformations $T = A_1 A_2 A_3 A_4 A_5 A_6$.

$$\begin{aligned} T &= T(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6) \\ \frac{\partial T}{\partial \theta_i} &= A_1 A_2 \cdots A_{i-1}(\theta_{i-1}) \frac{\partial A_i(\theta_i)}{\partial \theta_i} A_{i+1}(\theta_{i+1}) \cdots A_6(\theta_6) \\ \frac{\partial A_i(\theta_i)}{\partial \theta_i} &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} A_i(\theta_i) \\ \frac{\partial T}{\partial \theta_i} &= A_i(\theta_1) \cdots A_{i-1}(\theta_{i-1}) \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} A_i(\theta_i) \cdots A_6(\theta_6) \\ &= A_1(\theta_1) \cdots A_{i-1}(\theta_{i-1}) \left(\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0_2 & 0 \\ \hline & & 0_2 & 0_2 \end{array} \right) A_{i-1}(\theta_{i-1})^{-1} \cdots A_1(\theta_1)^{-1} A_1 A_2 A_3 A_4 A_5 A_6 \\ &= A_1(\theta_1) \cdots A_{i-1}(\theta_{i-1}) \underbrace{\left(\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0_2 & 0 \\ \hline & & 0_2 & 0_2 \end{array} \right)}_{\Delta^i} A_{i-1}(\theta_{i-1})^{-1} \cdots A_1(\theta_1)^{-1} T \\ \frac{\partial T}{\partial \theta_i} &= \Delta^i T \end{aligned}$$

To compute the Jacobian write:

$$\underbrace{\begin{pmatrix} 0 & -\delta_z & \delta_y & dx \\ \delta_z & 0 & -\delta_x & dy \\ -\delta_y & \delta_x & 0 & dz \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{\Delta} T = \sum_{i=1}^6 \frac{\partial T}{\partial \theta_i} d\theta_i = \sum_{i=1}^6 \Delta^i T d\theta_i$$

which implies:

$$\Delta = \sum_{i=1}^6 \Delta^i d\theta_i$$

Example 4-dof wrist

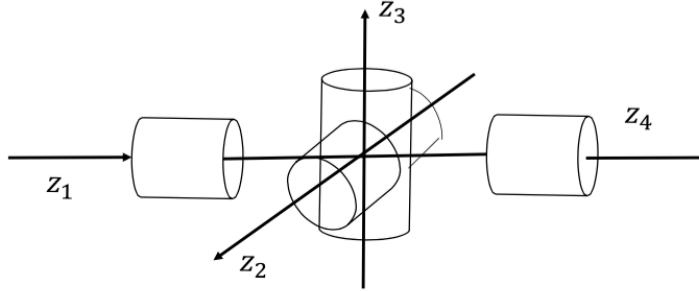


Figure 1: 4-dof wrist

The DH Parameters are:

i	α_i	a_i	d_i
1	90°	0	0
2	90°	0	0
3	90°	0	0
4	0	0	0

and the A_i and T matrices are:

$$\begin{aligned}
 A_1 &= \begin{pmatrix} \cos \theta_1 & -\sin \theta_1 & 0 \\ \sin \theta_1 & \cos \theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \\
 A_2 &= \begin{pmatrix} \cos \theta_2 & -\sin \theta_2 & 0 \\ \sin \theta_2 & \cos \theta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \\
 A_3 &= \begin{pmatrix} \cos \theta_3 & -\sin \theta_3 & 0 \\ \sin \theta_3 & \cos \theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \\
 A_4 &= \begin{pmatrix} \cos \theta_4 & -\sin \theta_4 & 0 \\ \sin \theta_4 & \cos \theta_4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
 T &= A_1 A_2 A_3 A_4 \sim \underbrace{\begin{pmatrix} 0 & -\delta_z & \delta_y \\ \delta_z & 0 & -\delta_x \\ -\delta_y & \delta_x & 0 \end{pmatrix}}_{\text{infinitesimals in tool space}}
 \end{aligned}$$

Compute the Jacobian (relative to the base frame):

$$\begin{aligned}
T &= \underbrace{\begin{pmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\tilde{A}_1} \underbrace{\begin{pmatrix} c_2 & s_2 & 0 \\ 0 & 0 & 1 \\ s_2 & -c_2 & 0 \end{pmatrix}}_{\tilde{A}_2} \underbrace{\begin{pmatrix} c_3 & 0 & s_3 \\ 0 & 1 & 0 \\ -s_3 & 0 & c_3 \end{pmatrix}}_{\tilde{A}_3} \underbrace{\begin{pmatrix} c_4 & -s_4 & 0 \\ s_4 & c_4 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\tilde{A}_4=A_4} \\
\frac{\partial T}{\partial \theta_1} &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} T \\
\frac{\partial T}{\partial \theta_2} &= \tilde{A}_1 \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tilde{A}_2 \tilde{A}_3 \tilde{A}_4 = \tilde{A}_1 \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tilde{A}_1^{-1} \underbrace{\tilde{A}_1 \tilde{A}_2 \tilde{A}_3 \tilde{A}_4}_T \\
&= \begin{pmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} T \\
&= \begin{pmatrix} 0 & 0 & -c_1 \\ 0 & 0 & -s_1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} T = \begin{pmatrix} 0 & 0 & -c_1 \\ 0 & 0 & -s_1 \\ c_1 & s_1 & 0 \end{pmatrix} T \\
\frac{\partial T}{\partial \theta_3} &= \tilde{A}_1 \tilde{A}_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \tilde{A}_2^{-1} \tilde{A}_1^{-1} T \\
&= \tilde{A}_1 \begin{pmatrix} 0 & c_2 & 0 \\ -c_2 & 0 & -s_2 \\ 0 & s_2 & 0 \end{pmatrix} \tilde{A}_1^{-1} T \\
&= \begin{pmatrix} s_1 c_2 & c_1 c_2 & s_1 s_2 \\ -c_1 c_2 & s_1 c_2 & -c_1 s_2 \\ 0 & s_2 & 0 \end{pmatrix} \begin{pmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} T \\
&= \begin{pmatrix} 0 & c_2 & s_1 s_2 \\ -c_2 & 0 & -c_1 s_2 \\ -s_1 s_2 & c_1 s_2 & 0 \end{pmatrix} T \\
\frac{\partial T}{\partial \theta_4} &= \tilde{A}_1 \tilde{A}_2 \tilde{A}_3 \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tilde{A}_3^{-1} \tilde{A}_2^{-1} \tilde{A}_1^{-1} T \\
&= \tilde{A}_1 \tilde{A}_2 \begin{pmatrix} 0 & -c_3 & 0 \\ c_3 & 0 & -s_3 \\ 0 & s_3 & 0 \end{pmatrix} \tilde{A}_2^{-1} \tilde{A}_1^{-1} T \\
&= \tilde{A}_1 \begin{pmatrix} 0 & -s_2 s_3 & c_3 \\ s_2 s_3 & 0 & -s_3 c_2 \\ -c_3 & s_3 c_2 & 0 \end{pmatrix} \tilde{A}_1^{-1} T \\
&= \underbrace{\begin{pmatrix} 0 & -s_2 s_3 & c_1 c_3 + s_1 c_2 s_3 \\ s_2 s_3 & 0 & s_1 c_3 - c_1 c_2 s_3 \\ -c_1 c_3 - s_1 c_2 s_3 & -s_1 c_3 + c_1 c_2 s_3 & 0 \end{pmatrix}}_{\text{skew symmetric}} T
\end{aligned}$$

Now

$$\begin{aligned} & \begin{pmatrix} 0 & -\delta_z & \delta_y \\ \delta_z & 0 & -\delta_x \\ -\delta_y & \delta_x & 0 \end{pmatrix} = \\ & \begin{pmatrix} 0 & -d\theta_1 & 0 \\ d\theta_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -c_1 d\theta_2 \\ 0 & 0 & -s_1 d\theta_2 \\ c_1 d\theta_2 & s_1 d\theta_2 & 0 \end{pmatrix} + \\ & \begin{pmatrix} 0 & c_2 d\theta_3 & s_1 s_2 d\theta_3 \\ -c_2 d\theta_3 & 0 & -c_1 s_2 d\theta_3 \\ -s_1 s_2 d\theta_3 & c_1 s_2 d\theta_3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & s_2 s_3 & c_1 c_3 + s_1 c_2 s_3 \\ s_2 s_3 & 0 & s_1 c_3 - c_1 c_2 s_3 \\ -c_1 c_3 - s_1 c_2 s_3 & -s_1 c_3 + c_1 c_2 s_3 & 0 \end{pmatrix} d\theta_4 \end{aligned}$$

Relate LHS to RHS entries and generate linear equations in the infinitesimals

$$\begin{pmatrix} \delta_x \\ \delta_y \\ \delta_z \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & s_1 & s_2 c_1 & -s_1 c_3 + c_1 c_2 s_3 \\ 0 & -c_1 & s_2 s_1 & c_1 c_3 + s_1 c_2 s_3 \\ 1 & 0 & -c_2 & s_2 s_3 \end{pmatrix}}_J \begin{pmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \\ d\theta_4 \end{pmatrix}$$

N. B. does not depend on θ_4

$$J = \begin{pmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & s_2 & c_2 s_3 \\ 0 & -1 & 0 & c_3 \\ 1 & 0 & -c_2 & s_2 s_3 \end{pmatrix}$$