ME/SE 740

Lecture 21

Kinematic Redundancy

Consider once again the 3-link planar manipulator:

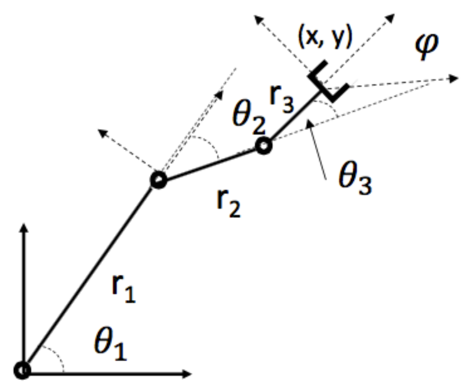

Figure 1: **Three Link Planar Manipulator**

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r_1 \cos\theta_1 + r_2 \cos(\theta_1+\theta_2) + r_3 \cos(\theta_1+\theta_2+\theta_3) \\ r_1 \sin\theta_1 + r_2 \sin(\theta_2+\theta_2) + r_3 \sin(\theta_1+\theta_2+\theta_3) \end{pmatrix}$$

If we are just interested in position of the origin of the tool frame, we have kinematic redundancy:

$$\begin{aligned} J &= \begin{pmatrix} -r_1 s_1 - r_2 s_{12} - r_3 s_{123} & -r_2 s_{12} - r_3 s_{123} & -r_3 s_{123} \\ r_1 c_1 + r_2 c_{12} + r_3 c_{123} & r_2 c_{12} + r_3 c_{123} & r_3 c_{123} \end{pmatrix} \\ &= \begin{pmatrix} c_1 & -s_1 \\ s_1 & c_1 \end{pmatrix} \underbrace{\begin{pmatrix} -r_2 s_2 - r_3 s_{23} & -r_2 s_2 - r_3 s_{23} & -r_3 s_{23} \\ r_1 + r_2 c_2 + r_3 c_{23} & r_2 c_2 + r_3 c_{23} & r_3 c_{23} \end{pmatrix}}_{C} \end{aligned}$$

J is non-singular if and only if CC^T is singular, if and only if $\sum_{i=1}^{3} (\det J_i)^2 = 0$, where J_i is the i^{th} 2×2 minor of C in the above expression.

$$\sum_{i=1}^{3}(\det J_i)^2 \quad = \quad r_1^2(r_2^2 s_2^2 + 2r_2 r_3 s_2 s_{23} + r_3^2 s_{23}^2) \; + \; r_3^2(r_1^2 s_{23}^2 + 2r_1 r_2 s_3 s_{23} + r_2^2 s_3^2) \; + \; r_2^2 r_3^2 s_3^2 \; = 0$$

$$\begin{aligned}
&\text{iff} \quad s_3 = 0 \quad \& \quad r_2 s_2 + s_3 s_{23} = 0 \quad \& \quad r_1 s_{23} + r_2 s_3 = 0 \\
&\text{iff} \quad s_2 = 0 \\
&\text{iff} \quad s_2 = s_3 = 0
\end{aligned}$$

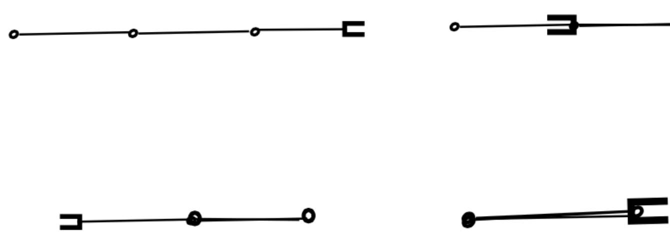

Figure 2: **4 Configurations Corresponding to Singularity**

Resolve velocity control for kinematically redundant mechanisms:

$$x \; = \; f(\theta) \quad \Longrightarrow \quad \dot{x} \; = \; J(\theta)\dot{\theta}$$

How do we solve if J is not square?

<u>Approach 1</u>

$$\dot{\theta} \; = \; J^{\dagger}\dot{x}$$

where $J^{\dagger} \; = \; J^T(JJ^T)^{-1}$, is the Moore-Penrose generalized inverse of J.

Suppose $A : \mathbb{R}^n \;\longrightarrow\; \mathbb{R}^m$ is a linear transformation (i.e., an $m \times n$ matrix) and $n \; > \; m$. Then there are infinitely many solutions to:

$$\underbrace{A}_{\text{given}}\underbrace{x}_{\text{find}} \quad = \quad \underbrace{y}_{\text{given}}$$

We can restrict the number of solutions by asking for the solution of "minimum norm," i.e., we solve:

$$\min \|x\|^2 \quad \text{subject to} \quad Ax = y$$

Using Lagrange multiplier thinking, find the critical point of:

$$\|x\|^2 \; + \; \lambda^T(Ax - y)$$

The critical point equations in vector form are:

$$\frac{\partial}{\partial x}(\|x\|^2 \ + \ \lambda^T(Ax - y)) \ = \ 2x + A^T\lambda \ = \ 0$$

now multiply the above equation on left by A which leads to:

$$2Ax + AA^T\lambda \ = \ 0, \quad \text{or} \quad 2y + AA^T\lambda \ = 0$$

Claim: If A has rank m then AA^T is invertible.

proof: A has rank m which implies $A^T x \ = \ 0 \iff x \ = \ 0$. Suppose that $AA^T x = 0$. Then $x^T AA^T x \ = \ 0$ and hence $\|A^T x\|^2 \ = \ 0$. Hence, $A^T x \ = \ 0$, which implies that $x \ = \ 0$. AA^T is a square matrix whose null space is the zero vector which means that AA^T is invertible.

Hence we can solve $2y + AA^T\lambda \ = 0$ for a unique value of λ

$$\lambda \ = \ -2(AA^T)^{-1}y$$

For "free space" motions (no obstacles) the inverse velocity solution is:

$$\dot{\theta} \ = \ J^\dagger \dot{x}$$

with $J^\dagger \ = \ J^T(JJ^T)^{-1}$ yields the minimizing $\dot{\theta}$ (minimizes $\|\dot{\theta}\|^2$) corresponding to $\dot{x}$, (Proposed by Daniel Whitney, 1969).

Problems

1. Klein-Huang (1983) *IEEE Transactions on Systems, Man and Cybernetics*, Vol. 13, showed the Moore-Penrose solution leads to a non-integrable relationship between joint space and tool space configurations. i. e., in general closed curves in tool space do not lead to closed curves in joint space,

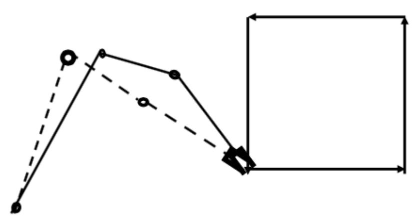

Figure 3: **Configurations Corresponding to Singularity**

and at the end of the move the joint are not in the same configuration (an undesirable characteristic).

2. Baillieul, Brocket, Hollerback (1984), (1984 CDC), showed that the Moore-Penrose inverse solution did not avoid kinematic singularities.

Demonstration

Let x_0 be any point in the tool space and let $\theta^\star$ be any point in a neighborhood of a singular configuration. Let $x^\star$ be the corresponding tool space point. Choose a workspace trajectory $x(\cdot)$, such that $x(0) \ = \ x^\star$ and $x(1) \ = \ x_0$ and let

$$\dot{\theta} \ = \ J^\dagger \dot{x} \qquad \text{(A)}$$

This will generate the corresponding joint space trajectory with:

$$f(\theta(1)) \; = \; x_0 \qquad \text{Call this} \;\; \theta_0 \; = \; \theta(1)$$

Imagine running the trajectory backwards. Consider a tool space trajectory $\tilde{x}(t) \; = \; x(1-t)$. This goes from x_0 to $x^\star$. Consider the joint space trajectory $\tilde{\theta}$ that this corresponds to via:

$$\dot{\tilde{\theta}} \; = \; J^\dagger(\tilde{\theta})\dot{\tilde{x}}, \qquad \text{with} \quad \tilde{\theta}(0) \; = \; \theta_0 \qquad (B)$$

Note that for the trajectory $\theta(\cdot)$ defined by (A):

$$\frac{d}{dt}\theta(1-t) \; = \; -\, J^\dagger(\theta(1-t))\dot{x}(1-t) \;\; = \;\; J^\dagger(\theta(1-t))\dot{\tilde{x}}$$

and by the uniqueness of solutions to ordinary differential equations:

$$\tilde{\theta}(t) \; = \; \theta(1-t), \quad \text{and in particular}$$

$$\tilde{\theta}(1) \; = \; \theta^\star$$

the nearly singular configuration we start with. Hence, since x_0 was arbitrary, we have shown that it cannot be assured apriori that the pseudo-inverse technique will generate trajectories that avoid singularities.

Other Approaches to Resolution of Redundancy

1. Imposition of a functional constraint: $h(\theta) = h(\theta_1, \ldots, \theta_n) = 0$

If there is one degree of redundancy, a single constraint will usually resolve it (an implicit function idea!). Care must be taken, however, not to winnow the workspace as seen in the example below. Constraining $h(\theta) = h(\theta_1, \ldots, \theta_n) = \theta_1$ to be zero, reduces the workspace from the large circle to the small circle as can be seen in the figure below.

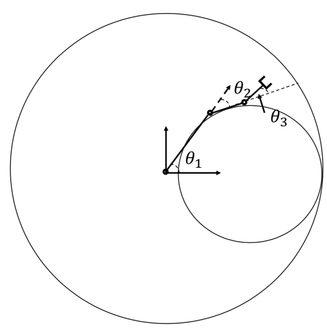

Figure 4: **Three link planar manipulator**

2. Optimize a criterion on configuration space (joint space variable, not velocities):

AT EACH POINT ALONG TRAJECTORIES, MAXIMIZE $g(\theta(t))$ SUBJECT TO $x(t) = f(\theta(t))$.

Examples of g's:

1. Distance to obstacles

2. Measures of dexterity (e.g., sum of squares of minors of the Jacobian = manipulatability index of Yoshikawa)

Derivations of necessary conditions:
1.

$$\frac{\partial g(\theta(t))}{\partial \theta} + \lambda^T J = 0$$

Let $\vec{n}_J$ be a vector in the null space of J. Then

$$\frac{\partial g(\theta(t))}{\partial \theta}\vec{n}_J + \lambda^T J \vec{n}_J = 0$$

implies that:

$$\frac{\partial g(\theta(t))}{\partial \theta}\vec{n}_J = 0$$

2. Let $\theta(t)$ be a "self-motion:"

$$x \equiv f(\theta(t))$$

(this is what it means to be a self-motion).

Then $J\dot{\theta} \equiv 0$. To maximize $g = g(\theta(t))$ with respect to t, we set $\frac{\partial g(\theta(t))}{\partial \theta}\dot{\theta} = 0$. But $\theta(t)$ is a self-motion, therefore, $\dot{\theta} \in \mathcal{N}(J)$ (the null space of J). Hence the necessary condition is:

$$\frac{\partial g(\theta(t))}{\partial \theta}\vec{n}_J = 0$$

Example

Consider the three-link manipulator depicted in Figure 4 above, with:

$$g(\theta_1, \theta_2, \theta_3) = sin^2\theta_2 + sin^2\theta_3$$

We then have:

$$\begin{aligned}
\frac{\partial g(\theta(t))}{\partial \theta} &= \begin{pmatrix} 0 \\ 2s_2c_2 \\ 2s_3c_3 \end{pmatrix} \\
J &= \begin{pmatrix} -s_1 - s_{12} - s_{123} & -s_{12} - s_{123} & -s_{123} \\ c_1 + c_{12} + c_{123} & c_{12} + c_{123} & c_{123} \end{pmatrix} \\
\vec{n}_J &\sim \begin{pmatrix} s_3 \\ -s_3 - s_{23} \\ s_2 + s_{23} \end{pmatrix} \\
\frac{\partial g(\theta(t))}{\partial \theta} \cdot \vec{n}_J &= s_3c_3s_2 + s_3c_3s_{23} - s_2c_2s_3 - s_2c_2s_{23} \\
&= 0
\end{aligned}$$

whenever $\theta_2 = \theta_3$.

Note: Mechanism can reach any point in the workspace and still have $\theta_2 = \theta_3$ satisfied!