Solutions

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ENG ME/SE 740:

Exercises (Set 3)

2 pts. 1. Prove that for any square matrices A and B that $e^{(A+B)t} = e^{At}e^{Bt} \iff [A,B]=0$.

2 pts 2. For the following 2×2 matrices, compute e^{At}:

$$\text{(a)}\quad A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \qquad \text{(b)}\quad A = \begin{pmatrix} -a & a \\ b & -b \end{pmatrix}$$

6 pts 3. Find θ, ω and v such that

$$\operatorname{Exp}\left[\begin{pmatrix} \hat{\omega} & v \\ 0 & 0 \end{pmatrix}\theta\right] = \begin{pmatrix} \frac{1+\sqrt{2}}{3} & \frac{2-\sqrt{2}\left(1+\sqrt{3}\right)}{6} & \frac{2+\sqrt{2}\left(-1+\sqrt{3}\right)}{6} & \frac{\sqrt{2}}{3}+\frac{\pi}{12} \\ \frac{2+\sqrt{2}\left(-1+\sqrt{3}\right)}{6} & \frac{1+\sqrt{2}}{3} & \frac{2-\sqrt{2}\left(1+\sqrt{3}\right)}{6} & \frac{-1}{3\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{6}}+\frac{\pi}{12} \\ \frac{2-\sqrt{2}\left(1+\sqrt{3}\right)}{6} & \frac{2+\sqrt{2}\left(-1+\sqrt{3}\right)}{6} & \frac{1+\sqrt{2}}{3} & \frac{-1}{3\sqrt{2}}-\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{6}}+\frac{\pi}{12} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

1. $$e^{(A+B)t} = I + (A+B)t + \tfrac{1}{2}(A+B)^2t^2 + \tfrac{1}{3!}(A+B)^3t^3 + \cdots$$

$$= I + (A+B)t + \tfrac{1}{2}(A^2 + AB + BA + A^2)t^2$$

$$+ \tfrac{1}{3!}\big(A^3 + A^2B + ABA + BA^2 + AB^2 + BAB + B^2A + B^3\big)t^3$$

$$+ \cdots$$

$$e^{At}e^{Bt} = \left(I + At + \tfrac{1}{2}A^2t^2 + \tfrac{1}{3!}A^3t^3 + \cdots\right)\left(I + Bt + \tfrac{1}{2}B^2t^2 + \tfrac{1}{3!}B^3t^3 + \cdots\right)$$

$$= I + At + Bt + \tfrac{1}{2}A^2t^2 + ABt^2 + \tfrac{1}{2}B^2t^2 + \tfrac{1}{3!}A^3t^3 + \tfrac{1}{2}A^2Bt^2 + \tfrac{1}{2}AB^2t^2 + \tfrac{1}{3!}B^3t^3.$$

The terms of all orders 2 and higher in $e^{(A+B)t}$ will have factors A, B appearing in all possible orders. The corresponding terms in $e^{At}e^{Bt}$ will be of the form A^kB^{n-k}. These can be equated $\Leftrightarrow$ $AB = BA$.

2(a) $$e^{\begin{pmatrix}\alpha & -\beta \\ \beta & \alpha\end{pmatrix}t} = \begin{pmatrix} e^{\alpha t}\cos\beta t & -e^{\alpha t}\sin\beta t \\ e^{\alpha t}\sin\beta t & e^{\alpha t}\cos\beta t \end{pmatrix}.$$

2(b)

$$e^{\begin{pmatrix}-a & a\\ b & -b\end{pmatrix}t}$$

$$= I - \frac{A}{a+b}\left(1 - (a+b)t + \frac{(a+b)^2t^2}{2} - \frac{(a+b)^3t^3}{3!} + \cdots\right) + \frac{A}{a+b}$$

where $A = \begin{pmatrix}-a & b\\ -b & a\end{pmatrix}$

$$= \begin{pmatrix} 1 + \frac{-a}{a+b} & \frac{a}{a+b}\\ \frac{b}{a+b} & 1 - \frac{b}{a+b}\end{pmatrix} - \frac{1}{a+b}\begin{pmatrix}-a & a\\ b & -b\end{pmatrix} e^{-(a+b)t}$$

$$= \begin{pmatrix} \frac{b}{a+b} + \frac{a}{a+b}e^{-(a+b)t} & \frac{a}{a+b} - \frac{a}{a+b}e^{-(a+b)t}\\ \frac{b}{a+b} - \frac{b}{a+b}e^{-(a+b)t} & \frac{a}{a+b} + \frac{b}{a+b}e^{-(a+b)t}\end{pmatrix}$$

3.

$$\exp\left(\begin{pmatrix}\hat{\omega} & v\\ 0 & 0\end{pmatrix}\right) = \begin{bmatrix} \frac{1+\sqrt{2}}{3} & \frac{2-\sqrt{2}\cdot(1+\sqrt{3})}{6} & \frac{2+\sqrt{2}\cdot(-1+\sqrt{3})}{6} & \frac{\sqrt{2}}{3}+\frac{\pi}{12} \\ \frac{2+\sqrt{2}\cdot(-1+\sqrt{3})}{6} & \frac{1+\sqrt{2}}{3} & \frac{2-\sqrt{2}\cdot(1+\sqrt{3})}{6} & \frac{-1}{3\cdot\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{6}}+\frac{\pi}{12} \\ \frac{2-\sqrt{2}\cdot(1+\sqrt{3})}{6} & \frac{2+\sqrt{2}\cdot(-1+\sqrt{3})}{6} & \frac{1+\sqrt{2}}{3} & \frac{-1}{3\cdot\sqrt{2}}-\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{6}}+\frac{\pi}{12} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\exp\left(\begin{pmatrix}\hat{\omega} & v\\ 0 & 0\end{pmatrix}\right) = \begin{bmatrix} e^{\hat{\omega}\cdot\theta} & \left[(I-R)\cdot\hat{\omega}+\left(\hat{\omega}^2+I\right)\cdot\theta\right]\cdot v \\ 0 & 1 \end{bmatrix}$$

$$R = e^{\hat{\omega}\cdot\theta} = \begin{bmatrix} 1-\left(\omega_2{}^2+\omega_3{}^2\right)\cdot(1-\cos(\theta)) & -\omega_3\cdot\sin(\theta)+\omega_1\cdot\omega_2\cdot(1-\cos(\theta)) & \omega_2\cdot\sin(\theta)+\omega_1\cdot\omega_3\cdot(1-\cos(\theta)) \\ \omega_3\cdot\sin(\theta)+\omega_1\cdot\omega_2\cdot(1-\cos(\theta)) & 1-\left(\omega_1{}^2+\omega_3{}^2\right)\cdot(1-\cos(\theta)) & -\omega_1\cdot\sin(\theta)+\omega_2\cdot\omega_3\cdot(1-\cos(\theta)) \\ -\omega_2\cdot\sin(\theta)+\omega_1\cdot\omega_3\cdot(1-\cos(\theta)) & \omega_1\cdot\sin(\theta)+\omega_2\cdot\omega_3\cdot(1-\cos(\theta)) & 1-\left(\omega_1{}^2+\omega_2{}^2\right)\cdot(1-\cos(\theta)) \end{bmatrix}$$

$$R := \begin{bmatrix} \frac{1+\sqrt{2}}{3} & \frac{2-\sqrt{2}\cdot(1+\sqrt{3})}{6} & \frac{2+\sqrt{2}\cdot(-1+\sqrt{3})}{6} \\ \frac{2+\sqrt{2}\cdot(-1+\sqrt{3})}{6} & \frac{1+\sqrt{2}}{3} & \frac{2-\sqrt{2}\cdot(1+\sqrt{3})}{6} \\ \frac{2-\sqrt{2}\cdot(1+\sqrt{3})}{6} & \frac{2+\sqrt{2}\cdot(-1+\sqrt{3})}{6} & \frac{1+\sqrt{2}}{3} \end{bmatrix} = \begin{pmatrix} 0.805 & -0.311 & 0.506 \\ 0.506 & 0.805 & -0.311 \\ -0.311 & 0.506 & 0.805 \end{pmatrix}$$

$$\theta := \operatorname{acos}\left(\frac{\operatorname{tr}(R)-1}{2}\right) = 0.785 = \frac{\pi}{4}$$

$$\omega_1 := \frac{\left(R_{3,2}-R_{2,3}\right)}{2\cdot\sin(\theta)} \to 0.577 \qquad \omega_2 := \frac{R_{1,3}-R_{3,1}}{2\cdot\sin(\theta)} \to 0.577 \qquad \omega_3 := \frac{R_{2,1}-R_{1,2}}{2\cdot\sin(\theta)} \to 0.577$$

$$\hat{\omega} := \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -0.577 & 0.577 \\ 0.577 & 0 & -0.577 \\ -0.577 & 0.577 & 0 \end{pmatrix} \qquad I := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Given

$$\begin{pmatrix} \frac{\sqrt{2}}{3}+\frac{\pi}{12} \\ \frac{-1}{3\cdot\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{6}}+\frac{\pi}{12} \\ \frac{-1}{3\cdot\sqrt{2}}-\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{6}}+\frac{\pi}{12} \end{pmatrix} = \left[(I-R)\cdot\hat{\omega}+\left(\hat{\omega}^2+I\right)\cdot\theta\right]\cdot\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\operatorname{find}\left(v_1, v_2, v_3\right) \to \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad v := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$