Solutions

Prof. J. Baillieul
Prof. T. Djaferis
Electrical & Computer
Engineering
Mechanical Engineering

(2/12/19)

ENG ME/SE 740

Exercises (Set 2) (Due 2/21/19)

1. Consider a planar manipulator with link lengths $r_1 = r_2 = 1$ and $r_3 = 1/2$. Find θ_1, θ_2, and θ_3 such that the end effector is at (1,1) with orientation $\phi = -45°$.

2 pts

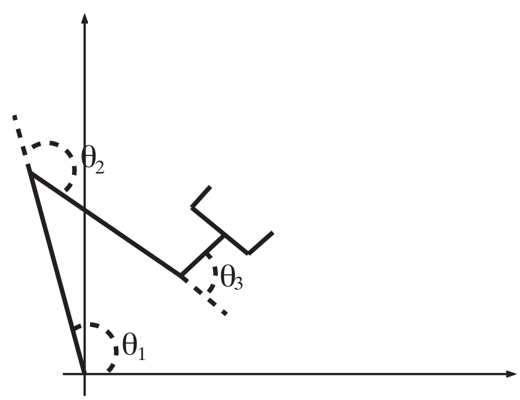


Figure 1: Mechanism of problem 1 but <u>not</u> in the configuration of the problem 1 statement.

2. We have seen a number of ways to describe the orientation of a rigid body with respect to a given coordinate frame. Consider the following rotation matrix:

6 pts

$$\begin{pmatrix} 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \end{pmatrix}.$$

Find the corresponding axis / angle parameters, the Euler-angles and the Tait-Bryan (pitch/roll/yaw) angles.

3. Which of the following subsets of $SE(3, \mathbb{R})$ are also subgroups?

2 pts

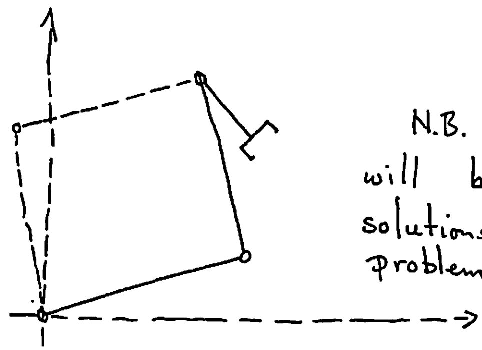
(a) $$\{\begin{pmatrix} \cos\theta & 0 & \sin\theta & x \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & z \\ 0 & 0 & 0 & 1 \end{pmatrix} : x, z \in \mathbb{R}; -\pi < \theta \le \pi\},$$

(b) $$\{\begin{pmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} : y, z \in \mathbb{R}; -\pi < \phi \le \pi\}$$

Justify your answer.

EXERCISE SET 2
SOLUTIONS

1.



N.B. There will be two solutions to this problem.

The general relation between joint angles and end effector position is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r_1 \cos\theta_1 & + r_2 \cos(\theta_1+\theta_2) & + r_3 \cos(\theta_1+\theta_2+\theta_3) \\ r_1 \sin\theta_1 & + r_2 \sin(\theta_1+\theta_2) & + r_3 \sin(\theta_1+\theta_2+\theta_3) \end{pmatrix}$$

The net rotation φ of the end effector is given by $\varphi = \theta_1+\theta_2+\theta_3$. This is $-45°$ in our problem, so that

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta_1 & + \cos(\theta_1+\theta_2) & + \frac{1}{2\sqrt{2}} \\ \sin\theta_1 & + \sin(\theta_1+\theta_2) & - \frac{1}{2\sqrt{2}} \end{pmatrix}$$

Letting $\bar{x} = x - 1/2\sqrt{2}$, $\bar{y} = y + 1/2\sqrt{2}$ we have

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} \cos\theta_1 & + \cos(\theta_1+\theta_2) \\ \sin\theta_1 & + \sin(\theta_1+\theta_2) \end{pmatrix} \qquad (0)$$

$$= \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} 1 + \cos\theta_2 \\ \sin\theta_2 \end{pmatrix}.$$

Because $\begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix}$ is an orthogonal matrix, the norm of $\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$ and $\begin{pmatrix} 1 + \cos\theta_2 \\ \sin\theta_2 \end{pmatrix}$ are the same. Hence

$$\bar{x}^2 + \bar{y}^2 = 2 + 2\cos\theta_2 \qquad (1)$$

Once θ_2 is determined from (1), θ_1 may be determined from (0) which we rewrite as

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} 1+\cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & 1+\cos\theta_2 \end{pmatrix} \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix}.$$

This may be solved for θ_1 by writing

$$\begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix} = \begin{pmatrix} 1+\cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & 1+\cos\theta_2 \end{pmatrix}^{-1} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$$

$$= \frac{1}{2+2\cos\theta_2} \begin{pmatrix} 1+\cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & 1+\cos\theta_2 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \qquad (2).$$

From (1) we write

$$\cos\theta_2 = \frac{\bar{x}^2+\bar{y}^2-2}{2}$$

$$= \frac{\left(1-\frac{1}{2\sqrt{2}}\right)^2 + \left(1+\frac{1}{2\sqrt{2}}\right)^2 - 2}{2}$$

$$= \frac{1}{8}$$

The corresponding <u>two</u> possible values for $\sin\theta_2$ are

$$\frac{3\sqrt{7}}{8}, \quad -\frac{3\sqrt{7}}{8}.$$

The corresponding two values for θ_2 are

$$\boxed{\theta_2 = \begin{matrix} 82.82^\circ \\ \text{or} \\ -82.82^\circ \end{matrix}}$$

From (2) the corresponding θ_1 values are computed as

$$\begin{pmatrix}\cos\theta_1\\ \sin\theta_1\end{pmatrix} = \frac{1}{2.25}\begin{pmatrix}1.125 & \frac{3\sqrt{7}}{8}\\ -\frac{3\sqrt{7}}{8} & 1.125\end{pmatrix}\begin{pmatrix}\frac{2\sqrt{2}-1}{2\sqrt{2}}\\ \frac{2\sqrt{2}+1}{2\sqrt{2}}\end{pmatrix}$$

$$\doteq \begin{pmatrix}.92008424\\ .39172053\end{pmatrix}$$

$$\theta_1 \sim 23.06^\circ \qquad (\text{corresp. to } \theta_2 \sim 82.82^\circ)$$

and

$$\begin{pmatrix}\cos\theta_1\\ \sin\theta_1\end{pmatrix} = \frac{1}{2.25}\begin{pmatrix}1.125 & -\frac{3\sqrt{7}}{8}\\ \frac{3\sqrt{7}}{8} & 1.125\end{pmatrix}\begin{pmatrix}\frac{2\sqrt{2}-1}{2\sqrt{2}}\\ \frac{2\sqrt{2}+1}{2\sqrt{2}}\end{pmatrix}$$

$$\doteq \begin{pmatrix}-.273637634\\ .96183285\end{pmatrix}$$

$$\theta_1 \sim 105.88^\circ \qquad (\text{corresp. to } \theta_2 \sim -82.82^\circ)$$

The corresponding values of θ_3 are easily computed, and the final solutions are

$$\boxed{\begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\end{pmatrix} = \begin{pmatrix}23.06^\circ\\ 82.82^\circ\\ -150.88^\circ\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\end{pmatrix} = \begin{pmatrix}105.88^\circ\\ -82.82^\circ\\ -68.06^\circ\end{pmatrix}}$$

$\boxed{2}$

The axis-angle representation is obtained using Rodrigues' formula:

$$\exp\left[\begin{pmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{pmatrix}\theta\right]$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -k_z & k_y \\ k_z & 0 & k_x \\ k_y & k_x & 0 \end{pmatrix} \sin\theta + \begin{pmatrix} -k_y^2 - k_z^2 & k_x k_y & k_x k_z \\ k_x k_y & -k_x^2 - k_z^2 & k_y k_z \\ k_x k_z & k_y k_z & -k_x^2 - k_y^2 \end{pmatrix} (1 - \cos\theta)$$

$$= \begin{pmatrix} k_x^2 (1-\cos\theta) + \cos\theta & -\sin\theta\, k_z + k_x k_y (1-\cos\theta) & \sin\theta\, k_y + k_x k_z (1-\cos\theta) \\ \sin\theta\, k_z + k_x k_y (1-\cos\theta) & k_y^2 (1-\cos\theta) + \cos\theta & -\sin\theta\, k_x - k_y k_z (1-\cos\theta) \\ -\sin\theta\, k_y + k_x k_z (1-\cos\theta) & \sin\theta\, k_x + k_y k_z (1-\cos\theta) & k_z^2 (1-\cos\theta) + \cos\theta \end{pmatrix}.$$

Setting this equal to

$$\begin{pmatrix} 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \end{pmatrix},$$

we sum the diagonal entries in both matrices. Recalling that $k_x^2 + k_y^2 + k_z^2 = 1$, this yields the simple relationship that

$$\cos\theta = 1/2.$$

Since we require $0 \le \theta \le \pi$, this means that $\theta = \pi/3 = 60°$. Then, subtracting on both sides the $(2,3)$ matrix entry from the $(3,2)$ matrix entry, we obtain

$$2\, k_x \sin\theta = \frac{2}{3} - (-\frac{1}{3}) = 1 \iff k_x = \frac{1}{2\sin\theta} = \frac{1}{\sqrt{3}}.$$

Similarly, $k_y = k_z = \frac{1}{\sqrt{3}}$.

To solve for the Euler angles we write

$$\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta\cos\varphi\cos\psi - \sin\varphi\sin\psi & -\cos\psi\sin\varphi - \cos\theta\cos\varphi\sin\psi & \cos\varphi\sin\theta \\ \cos\theta\cos\psi\sin\varphi + \cos\varphi\sin\psi & \cos\varphi\cos\psi - \cos\theta\sin\varphi\sin\psi & \sin\theta\sin\varphi \\ -\cos\psi\sin\theta & \sin\theta\sin\psi & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \end{pmatrix}.$$

By comparing (3,3) entries on both sides of this equation, $\cos\theta = \frac{2}{3}$. This has solution $\theta = 48.19°$ in the interval $[0, \pi]$. From this, we use the last column to obtain

$$\begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix},$$

from which we obtain $\varphi = -26.57°$. Comparing entries in the last row

$$\begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix},$$

from which we obtain $\psi = 63.43°$. Thus the Euler angles are given by

$$\begin{pmatrix} \varphi \\ \theta \\ \psi \end{pmatrix} = \begin{pmatrix} -26.57° \\ 48.19° \\ 63.43° \end{pmatrix}, \text{ and also } \begin{pmatrix} 153.43° \\ -48.19° \\ -116.57° \end{pmatrix},$$

the latter <u>not</u> satisfying $\theta \in [0, \pi]$.

To solve for the Tait-Bryan (RPY) angles we write

$$\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta\cos\varphi & \cos\varphi\sin\theta\sin\psi - \cos\psi\sin\varphi & \cos\varphi\cos\psi\sin\theta + \sin\varphi\sin\psi \\ \cos\theta\sin\varphi & \cos\varphi\cos\psi + \sin\theta\sin\varphi\sin\psi & \cos\psi\sin\theta\sin\varphi - \cos\varphi\sin\psi \\ -\sin\theta & \cos\theta\sin\psi & \cos\theta\cos\psi \end{pmatrix}$$

$$= \begin{pmatrix} 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \end{pmatrix}.$$

As above, we have

$$\begin{pmatrix} \varphi \\ \theta \\ \psi \end{pmatrix} = \begin{pmatrix} 45° \\ 19.47° \\ 45° \end{pmatrix}, \text{ and also } \begin{pmatrix} -135° \\ 160.53° \\ -135° \end{pmatrix},$$

3. In each case, it must be checked that the sets are closed under matrix multiplication and taking inverses

(a)

$$\begin{pmatrix} \cos\theta_1 & 0 & \sin\theta_1 & x_1 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_1 & 0 & \cos\theta_1 & z_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta_2 & 0 & \sin\theta_2 & x_2 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 & z_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta_1+\theta_2) & 0 & \sin(\theta_1+\theta_2) & x_2\cos\theta_1 + z_2\sin\theta_1 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta_1+\theta) & 0 & \cos(\theta_1+\theta_2) & -x_2\sin\theta_1 + z_2\cos\theta_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

✓

$$\begin{pmatrix} \cos\theta & 0 & \sin\theta & x \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & z \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta & -x\cos\theta + z\sin\theta \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & -x\sin\theta - z\cos\theta \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

✓

$\boxed{\therefore \text{(a) SPECIFIES A GROUP.}}$

(b)

$$\begin{pmatrix} \cos\varphi_1 & -\sin\varphi_1 & 0 & 0 \\ \sin\varphi_1 & \cos\varphi_1 & 0 & y_1 \\ 0 & 0 & 1 & z_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\varphi_2 & -\sin\varphi_2 & 0 & 0 \\ \sin\varphi_2 & \cos\varphi_2 & 0 & y_2 \\ 0 & 0 & 1 & z_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \cos(\varphi_1+\varphi_2) & -\sin(\varphi_1+\varphi_2) & 0 & -y_2\sin\varphi_1 \\ \sin(\varphi_1+\varphi_2) & \cos(\varphi_1+\varphi_2) & 0 & y_1 + \cos\varphi_1 y_2 \\ 0 & 0 & 1 & z_1+z_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\boxed{\therefore \text{The set in (b) is NOT a group.}}$