Solutions

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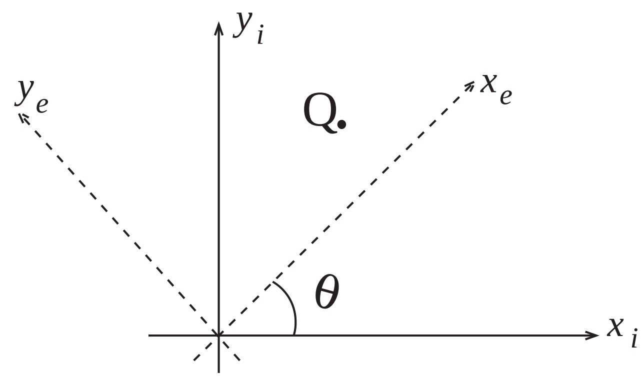
ENG MN 740:

Exercises (Set 1)

1. Referring to the figure below, the point Q may be expressed in terms of either coordinate system. Show that relation between the coordinates for Q in the two systems is given by

$$\begin{pmatrix} x_i \\ y_i \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_e \\ y_e \end{pmatrix}.$$

2 pts.



2 pts 2. Verify by a direct calculation that the group of rigid planar motions satisfies an associative law. (I.e. that for any three rigid motions $T_1,\ T_2,\ T_3$, we have $T_1 \circ (T_2 \circ T_3) = (T_1 \circ T_2) \circ T_3$.)

1 pt 3. Verify that the group of rigid planar motions is not Abelian. (Recall that a group is said to be Abelian if and only if $a \cdot b = b \cdot a$ for all a and b in the group.)

1 pt 4. A vector ${}^AP$ is rotated about $\hat{Z}_A$ by θ degrees and is subsequently rotated about $\hat{X}_A$ by ϕ degrees. Give the rotation matrix which will accomplish these rotations in the given order.

1 pt 5. A vector ${}^AP$ is rotated about $\hat{Y}_A$ by 30 degrees and is subsequently rotated about $\hat{X}_A$ by 45 degrees. Give the rotation matrix which will accomplish these rotations in the given order.

1 pt 6. A frame $\{B\}$ is located as follows: initially coincident with a frame $\{A\}$ we rotate $\{B\}$ about $\hat{Z}_B$ by θ degrees and then we rotate the resulting frame about $\hat{X}_B$ by ϕ degrees. Give the rotation matrix which will change the description of vectors from ${}^BP$ to ${}^AP$.

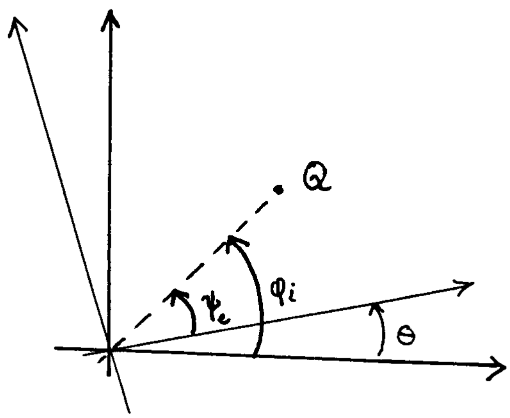
1 pt 7. A frame $\{B\}$ is located as follows: initially coincident with a frame $\{A\}$ we rotate $\{B\}$ about $\hat{Z}_B$ by 30 degrees and then we rotate the resulting frame about $\hat{X}_B$ by 45 degrees. Give the rotation matrix which will change the description of vectors from ${}^BP$ to ${}^AP$.

1 pt 8. ${}^A_BR$ is a 3×3 matrix with eigenvalues 1, $e^{+\alpha i}$, and $e^{-\alpha i}$, where $i = \sqrt{-1}$. What is the physical meaning of the eigenvector of ${}^A_BR$ associated with the eigenvalue 1?

MN740 EXERCISE SET 1 PROF. BAILLIEUL

SOLUTIONS

1.



Let $(\tilde{x}_e, \tilde{y}_e)$ be e-frame coordinates of Q. Let $(\tilde{x}_i, \tilde{y}_i)$ be i-frame coordinates of Q.

$$\cos\varphi_e = \frac{\tilde{x}_e}{\|Q\|} \qquad \sin\varphi_e = \frac{\tilde{y}_e}{\|Q\|}$$

length of vector from $P=\binom{x}{y}$ to Q

$$\cos\varphi_i = \frac{\tilde{x}_i}{\|Q\|} \qquad \sin\varphi_i = \frac{\tilde{y}_i}{\|Q\|}$$

Now $\varphi_i = \varphi_e + \theta$

$$\begin{pmatrix}\cos\varphi_i\\ \sin\varphi_i\end{pmatrix} = \begin{pmatrix}\cos(\varphi_e+\theta)\\ \sin(\varphi_e+\theta)\end{pmatrix} = \begin{pmatrix}\cos\varphi_e\cos\theta - \sin\varphi_e\sin\theta\\ \sin\varphi_e\cos\theta + \cos\varphi_e\sin\theta\end{pmatrix}$$

$$= \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\cos\varphi_e\\ \sin\varphi_e\end{pmatrix}$$

Mult. both sides of eq. by $\|Q\|$ to get

$$\begin{pmatrix}\tilde{x}_i\\ \tilde{y}_i\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\tilde{x}_e\\ \tilde{y}_e\end{pmatrix}$$

3. Let $g_1 = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & x_1 \\ \sin\theta_1 & \cos\theta_1 & y_1 \\ 0 & 0 & 1 \end{pmatrix}$,

$$g_2 = \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 & x_2 \\ \sin\theta_2 & \cos\theta_2 & y_2 \\ 0 & 0 & 1 \end{pmatrix}$$

Then

$$g_1 g_2 = \begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & x_1 + x_2\cos\theta_1 - y_2\sin\theta_1 \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & y_1 + x_2\sin\theta_1 + y_2\cos\theta_1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$g_2 g_1 = \begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & x_2 + x_1\cos\theta_2 - y_1\sin\theta_2 \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & y_2 + x_1\sin\theta_2 + y_1\cos\theta_2 \\ 0 & 0 & 1 \end{pmatrix}$$

In general, these two will not be equal (e.g. if $\theta_1 = 0$, $\theta_2 = \pi/4$, $x_1 = y_1 = 1$, $x_2 = y_2 = 0$)

4.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

5.

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ \frac{1}{2\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{\sqrt{3}}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} \end{pmatrix}$$

$$\begin{matrix} .866 & 0 & .5 \\ .354 & .707 & -.612 \\ -.354 & .707 & .612 \end{matrix}$$

6.

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$$

7.

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{1}{2} & \frac{\sqrt{3}}{2\sqrt{2}} & -\frac{\sqrt{3}}{2\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\begin{pmatrix} .866 & -.354 & .354 \\ .5 & .612 & -.612 \\ 0 & .707 & .707 \end{pmatrix}$$

8. As is well known (see any university text on linear algebra), because the matrix has distinct eigenvalues, it is diagonalizable. I.e. there is a basis with respect to which the matrix is written in the form

$$\begin{pmatrix} 1 & & \\ & e^{i\alpha} & \\ & & e^{-i\alpha} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha + i\sin\alpha & 0 \\ 0 & 0 & \cos\alpha - i\sin\alpha \end{pmatrix}.$$

The basis vectors may be chosen to be the eigenvectors.

Consider the change of basis

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & i/\sqrt{2} \\ 0 & i/\sqrt{2} & 1/2 \end{pmatrix}.$$

In the new basis, comprised of the eigenvector corresponding to 1 together with (complex) linear combinations of the other two eigenvectors, the matrix takes the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & i/\sqrt{2} \\ 0 & i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha + i\sin\alpha & 0 \\ 0 & 0 & \cos\alpha - i\sin\alpha \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -i/\sqrt{2} \\ 0 & -i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\alpha & s\alpha \\ 0 & -s\alpha & c\alpha \end{pmatrix}.$$

With respect to this basis, the matrix is clearly a rotation matrix. <u>The eigenvector corresponding to the eigenvalue 1 is aligned with the axis of rotation.</u>